A Text Book of **"Applied Mathematics III"** for S.E. Semester III of **"University of Mumbai"** written according to the Latest and Revised Syllabus of Computer, Electronics and Telecommunication Courses in force from June 2008

APPLIED MATHEMATICS - III

(Computer, Electronics & Telecommunication)

By

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Preface

I am pleased to place this up-to-date first edition of Applied Mathematics III in the hands of Second Year (Computer, Electronics and Telecommunication) Students of Engineering of University of Mumbai in particular and other Indian Universities in general. The book is written strictly according to the syllabus of **Applied Mathematics - III (Computer, Electronics and Telecommunication) of Mumbai University.** The book is based on my earlier book Applied Mathematics III and is modified according to the new syllabus. It contains a number of additional examples-solved and unsolved-taken mostly from recent examination papers. I have taken this opportunity to rewrite some old articles in some chapters so as to make the discussion complete and easier to understand. Also I have taken care to remove the printing mistakes that were brought to my notice by my colleagues.

The book is an outcome of over thirty-five years of teaching of Mathematics to students of Engineering and Science, of careful study of the syllabus of University of Mumbai, of clear understanding of the nature of question papers of more than fifteen years of Examinations of all branches of Engineering of University of Mumbai. I have also made extensive use of question papers of the past ten years of Pune, Shivaji and other Indian Universities so that after studying according to the present book, the students will have a clear understanding of the subject, of the type of examples they can expect at the university examination and will be hopefully able to solve them successfully.

Mathematics, no doubt, is a tool in the hands of an engineer. But to make it effective, it is imperative for him to know clearly how the tool works. This, in turn, requires clear understanding of the concepts and methods of higher mathematics. A teacher of Engineering Mathematics has to strike a golden mean between rigorous mathematical proofs of the theorems, which often tend to be tedious, and mere applications of these theorems to engineering problems which tend to be difficult to grasp. With this difficulty in mind, an attempt has been made in this book to explain the theory through illustrations, diagrams and examples whenever it was found sufficient for understanding the concepts involved. Some concepts and theorems are discussed at heuristic level also and rigorous mathematical proofs are given where it was found unavoidable. In short, every care has been taken to see that unnecessary material is not presented and necessary material is not left out.

To make the matter easy for understanding, main topics have been divided into sub-topics, and illustrative examples with complete solutions are given after theoretical discussion of each topic. Whenever possible, alternative methods of solving problems are given. A large number of examples is also given in the exercises. These examples are mostly taken from the university question papers and some are newly constructed according to the need. This will help students to know the nature of problems they are expected to face and the methods of solving such problems. In addition to this, a large number of miscellaneous examples with complete solutions is provided at the end of each chapter, and an equally large number of examples is given in the exercise for practice with sufficient hints. These examples are properly graded and carefully classified so that a student can have sufficient drill-work requiring a particular technique. In short, the book is so designed that it becomes complete in all respects and meets the demands of the students of both average and above-average calibre. It also fulfils the expectations of the university syllabus in letter and spirit.

I am extremely and sincerely thankful to Prof. A. N. Nakra; M.Sc. P. G. (BARC); M.Sc. (Nuclear Engg.) Canada, (Formerly Senior Scientist BARC) and Prof. Nelson Periera, M.Sc. (Pure Mathematics) for their continuous assistance, valuable guidance and constant encouragement in writing the whole series of books for Mumbai University. I also take this opportunity to express my sincere thanks to Prof. A. S. Desai, Prof. Mrs. Seema Latkar, Prof. A. B. Pawar, Prof. R. M. Pise, Prof. Mrs. S. Hegade, Prof. (Dr.) A. V. Dubewar, Prof. A. V. Deshmukh for their help and appreciation of my books.

I take this opportunity to express my sincere thanks and sense of gratitude to Shri Parimal J. Shah; M.A. who wholeheartedly accepted the responsibility of publishing the entire series. I also thank the printer for printing nicely and in time the book and the whole series. I am thankful to Shri. Rajan Bhate, Shri. Bhupendra Joshi and other members of the staff of C. Jamnadas and Co. for sparing no efforts to see that the books reach the students and teachers in time and for their cooperation and assistance in production and distribution of the series. My thanks are also due to Shri Anil M. Vhatkar of Mahalaxmi D.T.P. Center for neat, careful and swift type-setting and to Shri. Kaushal Kulkarni of Creative Concepts for beautiful cover design.

I hope that the students and teachers of Engineering Mathematics of Mumbai and other Universities will appreciate my efforts and will receive this edition enthusiastically. Any suggestions for enhancing the utility of the book and for removing errors that might have gone unnoticed will be gratefully acknowledged.

4th July, 2008

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Syllabus

APPLIED MATHEMATICS - III

S. E. Semester - III

(Mumbai University - Effective From June 2008)

Computer Engineering

Module 01 : Laplace Transforms

1. Function of bounded variation. Laplace transform of standard functions such as 1, t^a, e^{at}, sin at, cos at, sin $h\ at$, cos $h\ at$, erf (t). **(03)**
2. Linearity property of Laplace Transform, First Shifting property, Second Shifting property, Change of Scale property of L.T.

$$L\left\{t^n f(t)\right\},\ L\left\{\frac{f(t)}{t}\right\},\ L\left\{\int_0^t f(u)\,du\right\},\ L\left\{f^n(t)\right\}$$

Heaviside Unit step function, Dirac Delta function, Periodic functions and their Laplace Transforms. **(07)**

3. Inverse Lapalce Transform : Linearity property, use of theorems to find inverse Laplace Transform, Partial fractions method and convolution theorem (without proof). **(06)**
4. Applications to solve initial and boundary value problems involving ordinary differential equations with one dependent variable. **(03)**

Module 02 : Matrices

1. Types of Matrices. Adjoint of a matrix. Inverse of a matrix. Orthogonal matrix. Unitary matrix. Rank of a matrix. Reduction to a normal form PAQ. Linear dependence and independence of rows/columns over a field. **(07)**
2. System of homogeneous and non-homogeneous equation, their consistency and solutions. **(04)**

Module 03 : Fourier Series

1. Orthogonal and orthonormal set, Expressions of a function in a series of orthogonal functions. Dirichlet's condition. Fourier series of periodic function in the interval $[c, c + 2\pi]$, $[c, c + 2l\,]$. **(08)**
2. Dirichlet's theorem even and odd functions. Half range sine and cosine series. Parsevel's identities (without proof). **(04)**
3. Complex form of Fourier Series. **(02)**
4. Practical harmonic analysis. **(02)**

Module 04 : Fourier Transform

Introduction. Fourier integrals-Fourier sine and cosine integrals. Fourier sine and cosine transform. Linearity property. Change of scale property. Shifting property. Convolution theorem (without proof). **(06)**

Module 05 : Z-Transform

Z-transform of standard functions such as $Z(a^n)$, $Z(n^p)$. Linearity property. Damping rule. Shifting rules. Initial and Final value theorem. Convolution theorem (all without proof). Idea of Inverse Z-transform. **(06)**

Module 06 : Use of Scilab

(Computer Software) to solve integral transform. **(02)**

Term Work :

1. Based on above syllabus at least 10 tests assessed papers (10 marks).
2. One term test of 100 marks like university pattern must be conducted and scaled to 10 marks.
3. Attendance 05 marks.

Electronics and Telecommunication

Module 01 : Laplace Transforms **(10)**

1. Definition, linearity property, Laplace transform of standard functions 1, sin *at*, cos *at*, sin *h at*, cos *h at*.
2. First shifting theorem, Second shifting theorem, $L\{f(t)\}$, $L\{\ \}$. Change of scale property, $L\{\ \}$, $L\{\ \}$ (All theorems with proof). Convolution theorem (without proof).
3. Laplace transform of periodic functions, Error Function.
4. Heaviside Unit Step function and Dirac-delta function.

Module 02 : Laplace Transforms and Matrices **(12)**

1. Inverse Lapalce transforms, Solution of Ordinary differential equations using the Laplace Transform method
2. Types of Matrices-Symmetric, Skew-symmetric, Hermitian, Skew-Hermitian, Orthogonal and Unitary Matrices.
3. Inverse of a Matrix using Adjoint of a Matrix.

Module 03 : Matrices **(10)**

1. Echeloon form, Rank of a Matrix, Normal Form. PAQ in the Normal form.
2. System of Homogeneous and non-homogeneous equations, their consistency and solution using rank of a Matrix.
3. Linear Dependence and independence of vectors.
4. Solution of a system of simultaneous linear equations using Gauss elimination method, Gauss-Jordan reduction method, Gauss-Seidel iterative method.

Module 04 : Fourier Series **(12)**

1. Definition, Dirichlet's conditions (statement only). Fourier Series of functions with period.
2. Euler's formulae (with proof).
3. Fourier series of funtions having arbitrary period 2L.
4. Fourier series of odd and even functions.
5. Half range Fourier series, Parseval's identity (without proof).
6. Complex form of Fourier Series, Orthogonal and orthonormal functions.

Module 05 : Fourier Transforms **(08)**

1. Idea of Fourier Integral representation, Fourier Sine and Cosine Integral representation. Fourier Sine and Cosine Transforms. Linearity property, Change of Scale property. Shfiting property.
2. Convolution theorem (statement only) and related problems.

Module 06 : Z-transforms **(08)**

1. Sequence, Representation of a sequence. Basic operations on sequences. Definition of Z-transforms. Linearity property (without proof). Z-transforms of standard sequences - sin *k*, cos *k*, cos *hk*, sin *hk*, sin *k*, cos *k*.
2. Change of scale property. Shifting property. Inverse Z-transforms. Convolution theorem (statement only).
3. Inverse transform by Direct division. Binomial expansion and Partial fraction method.

Theory Examination

1. Question paper will be comprising of total 7 questions, each of 20 marks.
2. Only 5 questions need to be solved.

3. One question will be compulsory and based on entire syllabus.
4. Remaining questions will be mixed in nature (*e.g.* - suppose Q. 2 has part (a) from, module 3 then part (b) will be from any module other than module 3.)
5. In question paper weightage of each module will be proportional to number of respective lecture hours as mentioned in the syllabus.

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Z - Transforms

1. Introduction

We have already studied Laplace Transforms and Fourier Transforms. Both these transforms are continuous functions. These transforms are not useful for studying discrete systems. Linear systems in which the input signals are in the form of discrete pulses of short duration are called 'Linear Time Invariant' (LTI) systems. For the analysis of such systems we need Z-transforms. In this chapter we shall first get acquainted with sequences, then study Z-transforms and then inverse Z-transforms. After studying Laplace Transforms and Z-transforms, you will find that Z-transform is the discrete analogue of Laplace Transform. For every operational rule and application of Laplace transform there corresponds an operational rule or application of Z-transform. For example, you will find Linearity Property, Shifting Theorem, Convolution Theorem etc. in both Laplace Transforms and Z-transforms.

2. Sequences

If objects are arranged according to a certain rule, this arrangement is called a sequence. We are particularly interested in sequences whose members are real or complex numbers. So we define a sequence as follows.

Definition : An ordered set of real or complex numbers is called a **sequence**.

We shall denote a sequence by $\{ f(k) \}$ and k-th term of the sequence by $f(k)$. For example, we have a sequence

$$\{ 2^0, 2^1, 2^2, 2^3, \ldots\ldots 2^k, \ldots\ldots \}$$

For $k = 0$, $f(k) = 2^0$; for $k = 1$, $f(k) = 2^1$ ..... Thus, in a sequence we have to take into account, the order of a term k, the term of k-th order $f(k)$ and the set of all such ordered terms $\{ f(0), f(1), \ldots\ldots, f(k), \ldots\ldots \}$ called the sequence.

1. The most elementary way to denote a sequence is to list all the members of the sequence. For example,

$$\{f(k)\} = \{12, 10, \underset{\uparrow}{8}, 5, 3, 6, 9\}$$

The arrow $\uparrow$ indicates the element corresponding to $k = 0$. The elements on the left of the arrow correspond to $k = -1, -2, -3, \ldots\ldots$ and to the right correspond to $k = 1, 2, 3, \ldots\ldots$

2. Another way of denoting a sequence is to give the general term in terms of k which varies from $-\infty$ to ∞ taking integral values.

For example $\{f(k)\} = 2^k$ (where k is an integer). This sequence is

$$\{ \ldots\ldots, 2^{-3}, 2^{-2}, 2^{-1}, 2^0, 2^2, 2^3, \ldots\ldots \}$$

As illustrations we can have the following sequences and can have many more,

$$\{ 1, 1, 1, 1, \ldots\ldots \}$$
$$\{ 1, 2, 3, \ldots\ldots, k, \ldots\ldots \}$$
$$\{ 1^2, 2^2, 3^2, \ldots\ldots, k^2, \ldots\ldots \}$$
$$\left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots \right\}$$
$$\{ 1 \cdot 2^1, 2 \cdot 2^2, 3 \cdot 2^3, \ldots\ldots, k \cdot 2^k, \ldots\ldots \}$$
$$\{ 1 \cdot \alpha^1, 2 \cdot \alpha^2, 3 \cdot \alpha^3, \ldots\ldots, k \cdot \alpha^k, \ldots\ldots \}$$
$$\left\{ \frac{\alpha^1}{1}, \frac{\alpha^2}{2}, \frac{\alpha^3}{3}, \ldots\ldots, \frac{\alpha^k}{k}, \ldots\ldots \right\}$$
$$\left\{ \frac{5^1}{1!}, \frac{5^2}{2!}, \frac{5^3}{3!}, \ldots\ldots, \frac{5^k}{k!}, \ldots\ldots \right\}$$
$$\{ e^{1 \cdot \alpha}, e^{2 \cdot \alpha}, e^{3 \cdot \alpha}, \ldots\ldots, e^{k \cdot \alpha}, \ldots\ldots \}$$

3. Basic Operations On Sequences

We shall see below some properties of sequences through examples.

1. Addition : The sum (or difference) of two sequences is obtained by adding (or subtracting) the corresponding terms of the two sequences.

For example, if $\{f(k)\} = 1^3, 2^3, 3^3, 4^3, \ldots\ldots$

$\{g(k)\} = 1^2, 2^2, 3^2, 4^2, \ldots\ldots$

then
$$\{f(k)\} + \{g(k)\} = \{(1^3 + 1^2), (2^3 + 2^2), (3^3 + 3^2), \ldots\ldots, (k^3 + k^2) \ldots\ldots\}$$
$$= \{1^2 \cdot 2, 2^2 \cdot 3, 3^2 \cdot 4, \ldots\ldots, k^2 (k + 1) \ldots\ldots\}$$
$$\{f(k)\} - \{g(k)\} = \{(1^3 - 1^2), (2^3 - 2^2), (3^3 - 3^2), \ldots\ldots, (k^3 - k^2), \ldots\ldots\}$$
$$= \{1^2 \cdot 0, 2^2 \cdot 1, 3^2 \cdot 2, \ldots\ldots, k^2 \cdot (k - 1), \ldots\ldots\}$$

2. Scalar Multiplication : If α is a scalar then from a given sequence $f(k)$ we can obtain another sequence $\alpha \{f(k)\}$ by multiplying each term $\{f(k)\}$ of the sequence $\{f(k)\}$ by α.

For example, if $\{f(k)\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots$, then

$$3 \cdot \{f(k)\} = 3, \frac{3}{2}, \frac{3}{3}, \ldots\ldots, \frac{3}{k}, \ldots\ldots$$

If $\{f(k)\} = \sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots\ldots, \sqrt{k}, \ldots\ldots$, then

$$2 \cdot \{f(k)\} = 2\sqrt{1}, 2\sqrt{2}, 2\sqrt{3}, \ldots\ldots, 2\sqrt{k}, \ldots\ldots$$

3. Linearity : If α and β are two scalars then from two sequences $\{f(k)\}$ and $\{g(k)\}$, we can obtain another sequence by multiplying the terms of the two sequences by α and β as above and adding the corresponding terms.

i.e. $\alpha \{f(k)\} + \beta \{g(k)\} = \{\alpha \cdot f(k) + \beta \cdot g(k)\}$

For example, if $\{f(k)\} = 1, 2, 3, \ldots\ldots, k, \ldots\ldots$

and $\{g(k)\} = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots$

then, $2 \cdot \{f(k)\} + 3 \cdot \{g(k)\} = \{2 \cdot f(k) + 3 \cdot g(k)\}$

$$= 2(1) + 3\left(\frac{1}{1}\right), 2 \cdot 2 + 3\left(\frac{1}{2}\right), 2(3) + 3\left(\frac{1}{3}\right), \ldots\ldots, 2 \cdot k + 3\left(\frac{1}{k}\right) + \ldots\ldots$$
$$= 2 + \frac{3}{1}, 4 + \frac{3}{2}, 6 + \frac{3}{3}, \ldots\ldots, 2k + \frac{3}{k}, \ldots\ldots$$

4. Convergence And Divergence : Consider the following sequence

$$\frac{1+1}{1}, \frac{2+1}{2}, \frac{3+1}{3}, \frac{4+1}{4}, \ldots\ldots, \frac{k+1}{k}, \ldots\ldots$$

i.e. $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\ldots, 1 + \frac{1}{k}, \ldots\ldots$

i.e. 2, 1·5, 1·33, 1·25, ...

It is easy to see that as the number of terms become infinite the sequence goes on decreasing and ultimately takes the value 1. Such a sequence $\{f(k)\}$ is called a convergent sequence.

Definition : If $\{f(k)\}$ is a given sequence and if $f(k)$ tends to a (finite) real number L as k tends to infinity then $\{f(k)\}$ is called a **convergent sequence.**

The following sequences are convergent.

(i) $a, a, a, \ldots\ldots, a, \ldots\ldots, a, \ldots\ldots$ converges to a

(ii) $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{k}, \ldots\ldots$ converges to 0

(iii) $1 + \frac{1}{2^0}, 1 + \frac{1}{2^1}, 1 + \frac{1}{2^2}, \ldots\ldots, 1 + \frac{1}{2^k}, \ldots\ldots$ converges to 1.

Definition : A sequence which is not convergent *i.e.* which does not tend to a (finite) real number is called a **divergent** sequence.

The following are divergent sequences.

(i) 1, 2, 3, ..., k, ... diverges to ∞

(ii) $-1, -2, -3, \ldots\ldots, -k, \ldots\ldots$ diverges to $-\infty$

(iii) 1, 2, 1, 2, 1, 2, ... oscillates between 1 and 2

(iv) 0, 1, 0, 1, 0, 1, ... oscillates between 0 and 1.

EXERCISE

1. Write down the term corresponding to $k=3$ of the following sequence $\{-6, -3, -1, 0, \underset{\uparrow}{2}, 4, 6, 8, 10\}$ [**Ans. :** 8]

2. Write down the term corresponding to $k=-3$ of the following sequence. $\{-12, -10, -9, -7, \underset{\uparrow}{-5}, -3, 1, 4, 6, 10\}$ [**Ans. :** – 10]

3. Write down the sequence if k-th term is 3^k for $-2 \le k \le 4$.

   [**Ans. :** $\frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, 81$]

4. Write down the sequence where k-th term 2^k for $-\infty < k < \infty$.

   [**Ans. :** $\left\{\ldots\ldots, \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, \ldots\ldots\right\}$]

5. Write down the sequence where k-th term $= \begin{cases} 4^k, & k<0 \\ 3^k, & k \ge 0 \end{cases}$

   [**Ans. :** $\left\{\ldots\ldots, \frac{1}{64}, \frac{1}{16}, \frac{1}{4}, 1, 3, 9, 27, \ldots\ldots\right\}$]

6. Write down the sequence where k-th term $= \begin{cases} a^k, & k<0 \\ b^k, & k \ge 0 \end{cases}$

   [**Ans. :** $\left\{\ldots\ldots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a}, 1, b, b^2, b^3, \ldots\ldots\right\}$]

4. Z - transforms

We shall now define Z-transform of a sequence.

Definition : Let $\{f(k)\} = \{\ldots\ldots f(-3), f(-2), f(-1), f(0), f(1), f(2), f(3), \ldots\ldots\}$ be a sequence of terms where k varies from $-\infty$ to ∞.

Let $z = x + iy$ be a complex number then

$$Z\{f(k)\} = \ldots\ldots + f(-3)\, z^3 + f(-2)\, z^2 + f(-1)\, z^{-1} + f(0)\, z^0 + f(1)\, z^{-1} + f(2)\, z^{-2} + f(3)\, z^{-3} + \ldots\ldots$$

$$= \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{\infty} \frac{f(k)}{z^k}$$

is called the **Z-transform of the sequence $\{f(k)\}$.**

Thus, $$\boxed{Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k}}$$

Notation : Unfortunately there is no unanimity in the notations used in the case of Z-transform. Some use $x_k, y_k, \ldots\ldots$ to denote sequences and $x_0, x_1, \ldots\ldots, y_0, y_1, \ldots\ldots$ to denote the terms of the sequences. We shall denote the sequences by $\{f(k)\}, \{g(k)\}, \ldots\ldots$ the terms of the sequences by $f(1), f(2), \ldots\ldots, g(1), g(2), \ldots\ldots$ and Z-transforms by $Z\{f(k)\}, Z\{g(k)\}, \ldots\ldots$ or by $F(z), G(z), \ldots\ldots$ etc.

Notes

1. It is necessary to know which is the zeroth term, first term, second term .... minus first term, minus second term ..... *i.e.* we must know the order of each term. To obtain Z-transform of a sequence we multiply each term by **negative power of z of the order of that term** and take the sum.

2. $Z\{f(k)\}$ is a function of a complex variable z and is defined only if the sum is finite *i.e.* **if the infinite series $\Sigma\, f(k)\, z^{-k}$ is absolutely convergent.** We shall denote the Z-transform of the sequence $\{f(k)\}$ by $Z\{f(k)\}$ or by $F(z)$.

3. Wherever necessary we shall denote the sequences by $\{f(k)\}, \{g(k)\}$ etc.

4. If Z-transform of $\{f(k)\}$ is $F(z)$ we call **$\{f(k)\}$ the inverse Z transform of $F(z)$ and denote** it by $Z^{-1}[F(z)]$.

Ex. 1 : If $\{f(k)\} = \{-6, -3, \underset{\uparrow}{0}, 2, 4\}$,

find $Z\{f(k)\}$ where $\uparrow$ denotes the element corresponding to $k = 0$.

Sol. : $Z\{f(k)\} = \Sigma f(k)\, z^{-k}$

$$= f(-2)\, z^{-2} + f(-1)\, z^{-1} + f(0)\, z^0 + f(1)\, z^1 + f(2)\, z^2$$

where, $f(-2) = -6,\ f(-1) = -3,\ f(0) = 0,\ f(1) = 2,\ f(2) = 4.$

$$\therefore\quad Z\{f(k)\} = \sum_{k=-2}^{2} f(k)\, z^{-k}$$

$$= (-6)\, z^2 + (-3)\, z^1 + 0 \cdot z^0 + 2z^{-1} + 4z^{-2}$$

$$= -6z^2 + 3z + 0 + \frac{2}{z} + \frac{4}{z^2}$$

Ex. 2 : If $\{f(k)\} = \{9, 6, \underset{\uparrow}{3}, 0, -3, -6, -9\}$, find $Z\{f(k)\}$.

Sol. : Since 3 is the term corresponding to $k = 0$. We have

$f(-2) = 9,\ f(-1) = 6,\ f(0) = 3,\ f(1) = 0,$
$f(2) = -3,\ f(3) = -6,\ f(4) = -9.$

$$\therefore\quad Z\{f(k)\} = \sum_{k=-2}^{4} f(k)\, z^{-k}$$

$$= 9z^2 + 6z^1 + 3z^0 + 0z^{-1} - 3z^{-2} - 6z^{-3} - 9z^{-4}$$

$$\therefore \quad Z\{f(k)\} = 9z^2 + 6z + 3 + 0 - \frac{3}{z^2} - \frac{6}{z^3} - \frac{9}{z^4}.$$

Ex. 3 : If $\{f(k)\} = \{2^0, 2^1, 2^2, 2^3, ...\}$ find $Z\{f(k)\}$.

Sol. : Z-transform of the sequence is

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{k=\infty} f(k)\, z^{-k}$$

$$= 2^0 z^0 + 2^1 z^{-1} + 2^2 z^{-2} + 2^3 z^{-3} + ...$$

$$= 1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + ...$$

$$= \frac{1}{1-(2/z)} = \frac{z}{z-2} \quad \text{if } \left|\frac{2}{z}\right| < 1$$

$$\left(\because \; s_\infty = a + ar + ar^2 + ... = \frac{a}{1-r}, \text{ if } |r| < 1\right)$$

Ex. 4 : If $\{f(k)\} = \begin{cases} 4^k, & \text{for } k < 0 \\ 3^k, & \text{for } k \geq 0 \end{cases}$, find $Z\{f(k)\}$.

Sol. : The sequence is

$$\{f(k)\} = \{....., 4^{-4}, 4^{-3}, 4^{-2}, 4^{-1}, 3^0, 3^1, 3^2, 3^3, ...\}$$

And Z-transform of $f(k)$ is

$$\therefore \quad Z\{f(k)\} = \{....., 4^{-4} z^4 + 4^{-3} z^3 + 4^{-2} z^2 + 4^{-1} z + 3^0 z^0 + 3^1 z^{-1} + 3^2 z^{-2} + 3^3 z^{-3} + ...\}$$

We write positive powers of z in reverse order.

$$\therefore \quad Z\{f(k)\} = \left[\frac{z}{4} + \frac{z^2}{4^2} + \frac{z^3}{4^3} + ...\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + ...\right]$$

$$= \frac{z}{4}\left[1 + \frac{z}{4} + \left(\frac{z}{4}\right)^2 + ...\right] + \left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + ...\right]$$

$$= \frac{z}{4} \cdot \frac{1}{1-(z/4)} + \frac{1}{1-(3/z)} \quad \text{if } \left|\frac{z}{4}\right| < 1, \left|\frac{3}{z}\right| < 1.$$

$$= \frac{z}{4} \cdot \frac{4}{4-z} + \frac{z}{z-3} = \frac{z}{4-z} + \frac{z}{z-3}$$

$$= \frac{z}{(4-z)(z-3)} \quad \text{if } 3 < |z| < 4.$$

Note : We shall require the following results in finding Z-transforms.

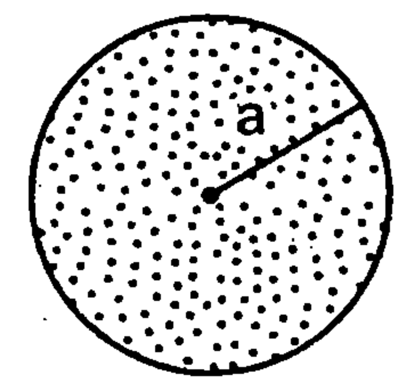
1. $1 + r + r^2 + r^3 + ... = \dfrac{1}{1-r}$ if $|r| < 1$.

2. $1 - x + x^2 - x^3 + ... = (1+x)^{-1}$ if $|x| < 1$.
3. $1 + x + x^2 + x^3 + ... = \sum_{k=0}^{\infty} x^k = (1-x)^{-1}$ if $|x| < 1$.
4. $1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(x-2)}{3!}x^3 + ... = (1+x)^n$.
5. $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ... = e^x$
6. If $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$.
7. **The set $|z| < a$.**

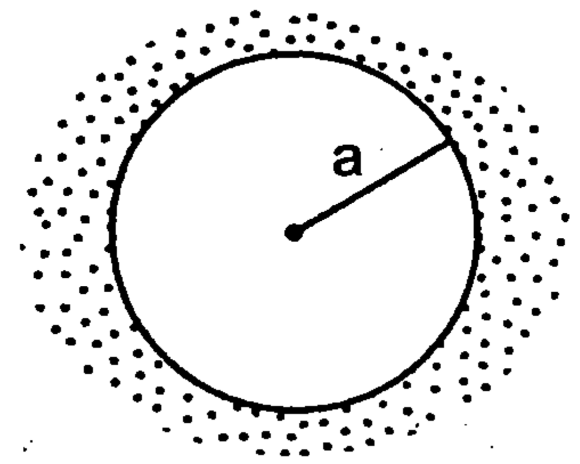
Since $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$.

$\therefore |z| < a$ means $\sqrt{x^2 + y^2} = a$ *i.e.* $x^2 + y^2 < a^2$.

This means, $|z| < a$ is the set of points inside the circle of radius a and centre at the origin. By the same reasoning $|z| > a$ is the set of points outside the circle with radius a and centre at the origin.



$|z| < a$



$|z| > a$

EXERCISE

1. Write down the Z-transforms of the following sequences
   (i) $\{f(k)\} = \{8, 6, 4, 2, 0, 1, 3, 5, 7\}$ (with $\uparrow$ under 0)

   (ii) $\{f(k)\} = \{-6, -4, -2, 1, 2, 4, 6\}$ (with $\uparrow$ under 1)

   [**Ans. : (i)** $8z^6 + 6z^5 + 4z^4 + 2z^3 + 0 + 1z + 3 + \dfrac{5}{z} + \dfrac{7}{z^2}$

   (ii) $-6z^3 - 4z^2 - 2z + 1 + \dfrac{2}{z} + \dfrac{4}{z^2} + \dfrac{6}{z^3}$]

2. Write down the Z-transform of the following sequences
   (i) $\{f(k)\} = 3^k,\; k \geq 0$ (ii) $\{f(k)\} = 5^k,\; k \geq 0$

   [**Ans. : (i)** $\dfrac{z}{z-3}, \left|\dfrac{3}{z}\right| < 1$, **(ii)** $\dfrac{z}{z-5}, \left|\dfrac{5}{z}\right| < 1$.]

6. Inverse Z-Transform

Definition : If $F(z)$ is the Z-transform of the sequence $\{f(k)\}$ then the sequence $\{f(k)\}$ is called the inverse Z-transform of $F(z)$ and is denoted as

$$\boxed{f(k) = Z^{-1}[F(z)]}$$

Thus, we have if $Z\{f(k)\} = F(z)$, then $\{f(k)\} = Z^{-1}[f(z)]$ and vice versa.

7. Region of Convergence (ROC)

We shall try to understand this important concept in relation to Z-transforms through an example.

Consider the following sequence

$$f(k) = 0 \quad \text{for } k < 0$$
$$= 4^k \quad \text{for } k \geq 0$$

i.e. the sequence $\{f(k)\} = \{4^0, 4^1, 4^2, 4^3, \ldots\ldots, 4^k, \ldots\ldots\}$

Its Z-transform by definition is

$$Z\{f(k)\} = \Sigma f(k) z^{-k}$$
$$= \sum_{k=0}^{\infty} 4^k z^{-k}$$
$$= 4^0 z^0 + 4z^{-1} + 4^2 z^{-2} + 4^3 z^{-3} + \ldots.$$
$$= 1 + \frac{4}{z} + \frac{16}{z^2} + \frac{64}{z^3} + \ldots..$$

Notice that $Z\{f(k)\}$ is a Geometric Progression with common ration $4/z$. We know that the sum S of infinite terms of a G.P. with first term 1 and common ratio r is given by $S = \frac{1}{1-r}$ if $|r| < 1$.

The sum of the above series *i.e.* the Z-transform is

$$Z\{f(k)\} = \frac{1}{1-(4/z)}$$
$$= \frac{z}{z-4}, \text{ if } \left|\frac{4}{z}\right| < 1$$

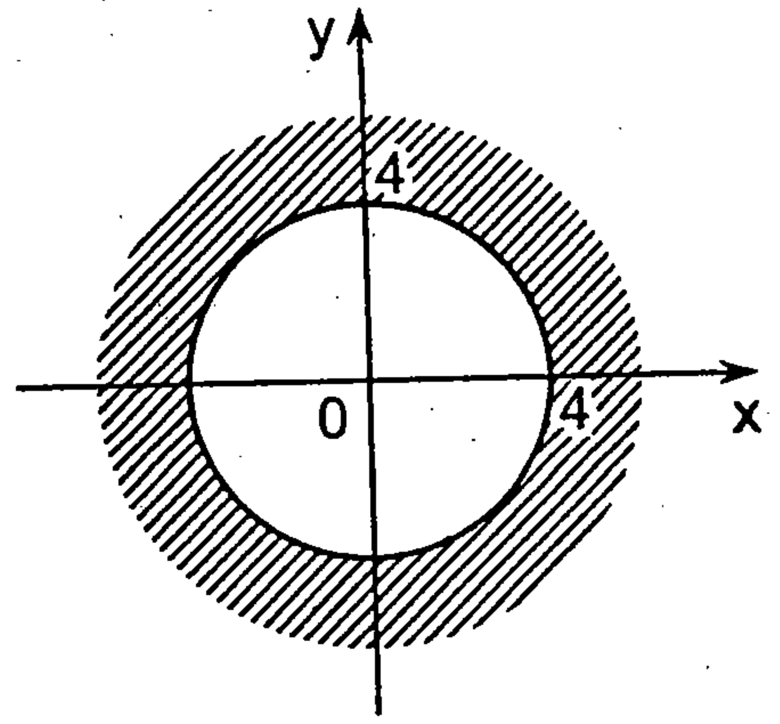
i.e. $\quad 4 < |z| \quad$ *i.e.* $\quad |z| > 4$.

Since, for the Z-transform to exist the corresponding series must be convergent. The above Z-transform is defined only if $|z| > 4$.

[**Note :** Note that a G.P. $a, ar, ar^2, \ldots.. ar^n \ldots..$ is convergent if $|r| < 1$ and its sum

$$S = \frac{a}{1-r} \text{ where } |r| < 1.]$$

But $|z| = 4$ is a circle with centre at the origin and radius = 4. Hence, the above Z-transform is defined if $z > 4$ *i.e.* if z is on the exterior of the circle $|z| = 4$.



The region for which $\Sigma f(k) z^{-k}$ is convergent is called the region of convergence denoted in short by R.O.C.

Ex. : Find the Z-transform and the region of convergence of

$$f(k) = 5^k \quad \text{for } k < 0$$
$$= 3^k \quad \text{for } k \geq 0.$$

Sol. : By definition $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

$$\therefore \quad Z\{f(k)\} = \sum_{k=-\infty}^{-1} 5^k z^{-k} + \sum_{k=0}^{\infty} 3^k z^{-k}$$

Putting $k = -n$ in the first series, we get

$$Z\{f(k)\} = \sum_{n=1}^{\infty} 5^{-n} z^n + \sum_{k=0}^{\infty} 3^k z^{-k}$$

$$Z\{f(k)\} = \left[\frac{z}{5} + \frac{z^2}{5^2} + \frac{z^3}{5^3} + \ldots..\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots..\right]$$

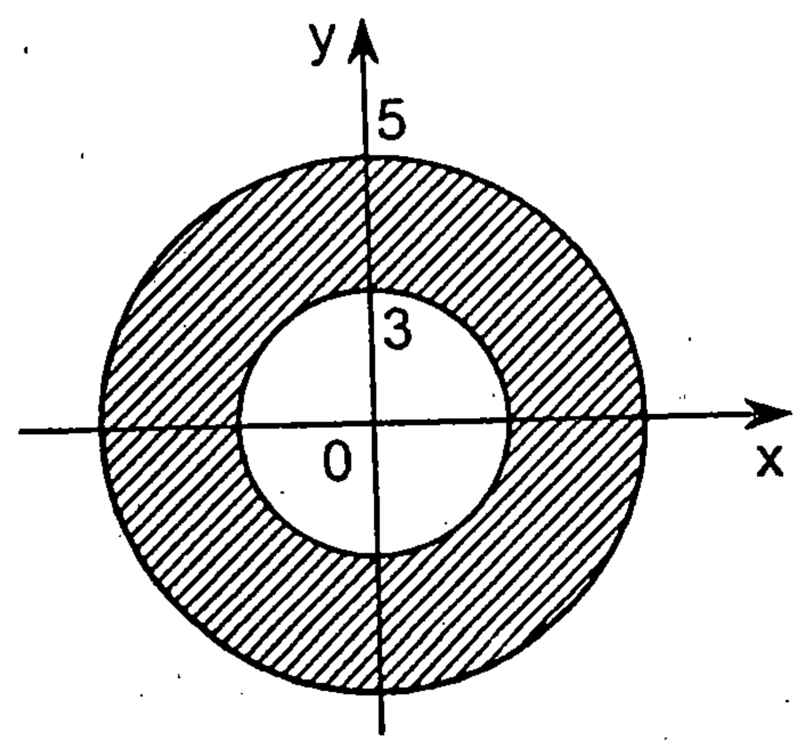
$$= \frac{z}{5}\left[1 + \frac{z}{5} + \frac{z^2}{5^2} + \ldots..\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \ldots..\right]$$

$$= \frac{z}{5} \cdot \frac{1}{1-(z/5)} + \frac{1}{1-(3/z)}$$

$$= \frac{z}{5-z} + \frac{z}{z-3} = \frac{2z}{(5-z)(z-3)}$$

Now, $Z\{f(k)\}$ is the sum of two Geometric Progressions with the common ratios $(z/5)$ and $(3/z)$ respectively. The series will be convergent if $|z/5| < 1$ and $|3/z| < 1$. *i.e.* $|z| < 5$ and $3 < |z|$ *i.e.* $3 < |z| < 5$.

But $|z| = 3$ is a circle with centre at the origin and radius 3 and $|z| = 5$ is a circle with centre at the origin and radius 5. Hence, $Z\{f(k)\}$ is convergent if z lies between the annulus as shown in the figure. This is the region of convergence of $Z\{f(k)\}$ which is shown by shaded area.



$\therefore \quad$ ROC is $3 < |z| < 5$.

8. Z-Transforms of Some Standard Functions

Ex. 1 : Find the Z-transform of *Unit Impulse function*

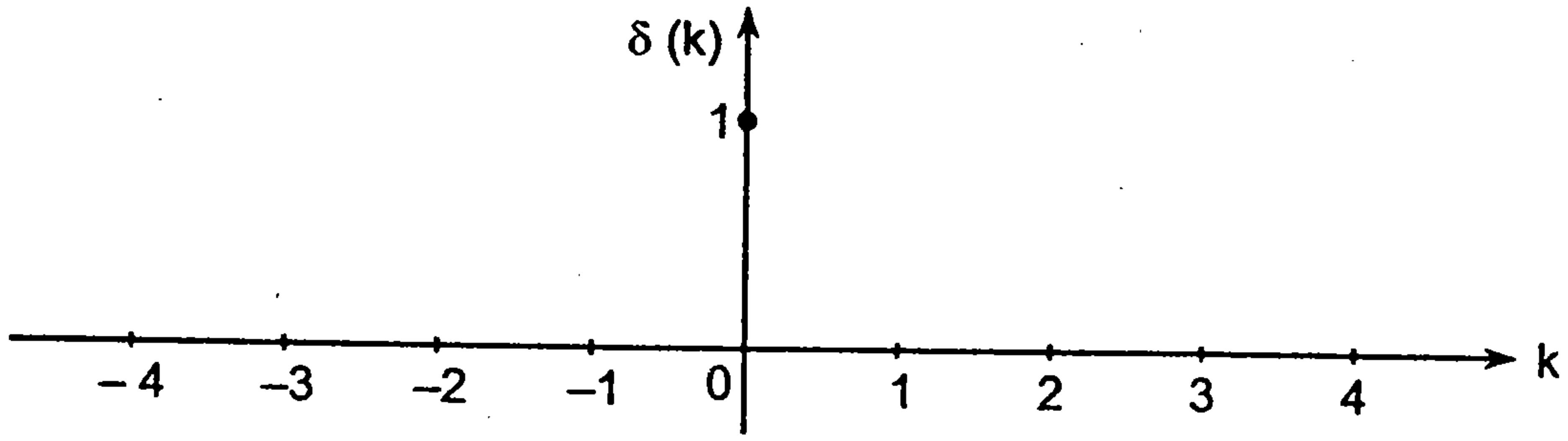
$$\delta(k) = 1 \text{ for } k = 0$$
$$= 0 \text{ otherwise}$$

Sol. : $Z\{\delta(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$

$$= \{.....0+0+0+1 \cdot z^0 + 0+0+0.....\}$$
$$= 1 \text{ for all } z$$

This is convergent for all z $\therefore$ ROC is whole of z-plane.

The graph of unit impulse function is,



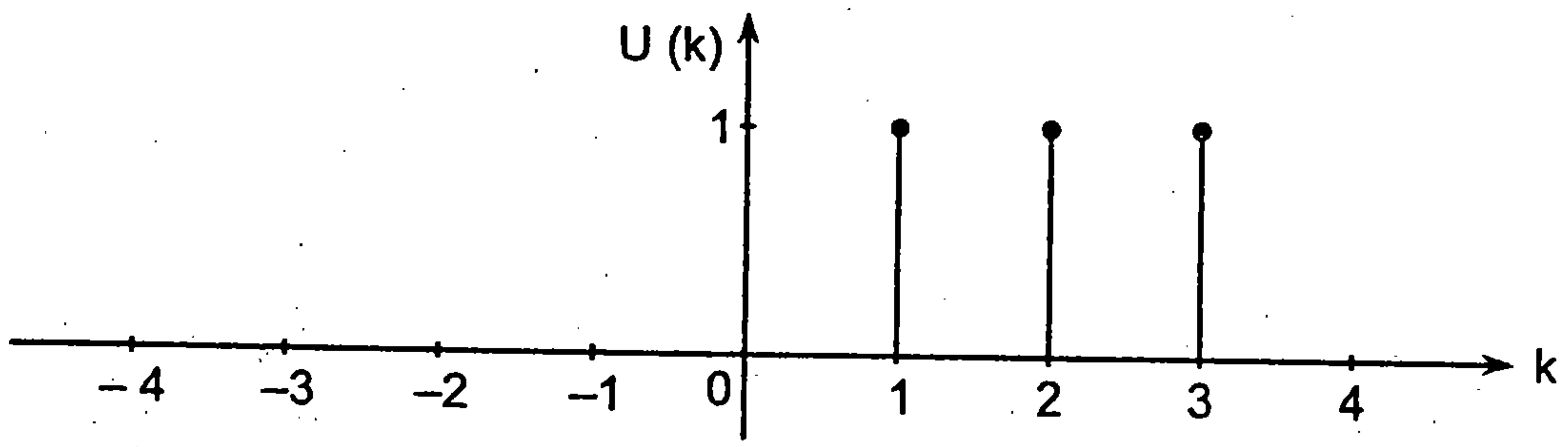
Ex. 2 : Find the Z-transform of *Discrete Unit Step function*

$$U(k) = 1 \text{ for } k \geq 0$$
$$= 0 \text{ for } k < 0$$

Sol. : $$Z\{U(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} 1 \cdot z^{-k} = \sum_{k=0}^{\infty} z^{-k}$$

$$= \left[1 + \frac{1}{z} + \frac{1}{z^2} + ...\right] = \frac{1}{1-(1/z)} = \frac{z}{z-1}$$

The graph of discrete unit step function is,



This is convergent if $|1/z| < 1$ *i.e.* $1 < |z|$ *i.e.* $|z| > 1$

$\therefore$ ROC is $|z| > 1$.

Since, $Z\{U(k)\} = \dfrac{z}{z-1}$, $\quad Z^{-1}\left[\dfrac{z}{z-1}\right] = \{U(k)\}$

where Z^{-1} denotes inverse Z-transform.

Ex. 3 : Find the Z-transform of $f(k) = k\,\alpha^k$, $k \geq 0$.

Sol. : Assuming $f(k) = 0$ for $k < 0$,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^k + \sum_{k=0}^{\infty} k\,\alpha^k z^{-k}$$

$$= 0 + 1 \cdot \frac{\alpha}{z} + 2 \cdot \frac{\alpha^2}{z^2} + 3 \cdot \frac{\alpha^3}{z^3} + ... \qquad ....(1)$$

$$= \frac{\alpha}{z}\left(1 + 2\frac{\alpha}{z} + 3\frac{\alpha^2}{z^2} + ...\right)$$

$$= \frac{\alpha}{z}\left(1 - \frac{\alpha}{z}\right)^{-2} = \frac{\alpha}{z} \cdot \frac{1}{[1-(\alpha/z)]^2} = \frac{\alpha z}{(z-\alpha)^2}$$

Applying D'Alembert's radio test to (1), we find that the series is convergent if $|\alpha/z| < 1$ *i.e.* $|\alpha| < |z|$.

$\therefore$ ROC is $|z| > |\alpha|$

$$\because \quad Z\{k\,\alpha^k\} = \frac{\alpha z}{(z-\alpha)^2}, \quad Z^{-1}\left[\frac{\alpha z}{(z-\alpha)^2}\right] = k\,\alpha^k$$

Particular Cases : (i) Find the Z-transform of $f(k) = k2^k$, $k \geq 0$.

(ii) Find the z-transform of $f(k) = k\,2^k + k\,3^k$.

Sol. : Put (i) $\alpha = 2$ in the above example and (ii) $\alpha = 2$, $\alpha = 3$,

[**Ans. : (i)** $\dfrac{2z}{(z-2)^2}$, $|z| > 2$, **(ii)** $\dfrac{2z}{(z-2)^2} + \dfrac{3z}{(z-3)^2}$, $|z| > 3$]

Ex. 4 : Find the z-transform of $f(k) = \alpha^k$, $\alpha > 0$, $k \geq 1$.

Sol. : We have

$$z\{f(k)\} = \sum_{k=1}^{\infty} \alpha^k z^{-k} = \frac{\alpha}{z} + \frac{\alpha^2}{z} + \frac{\alpha^3}{z^3} + ...$$

$$= \frac{\alpha}{z}\left(1 + \frac{\alpha}{z} + \frac{\alpha^2}{z^2} + ...\right) \qquad [\text{ G.P. }]$$

$$= \frac{\alpha}{z} \cdot \frac{1}{1-(\alpha/z)} = \frac{\alpha}{z-\alpha}, \; |\alpha| < |z|.$$

Ex. 5 : Find the Z-transform of $f(k) = \dfrac{\alpha^k}{k}$, $k \geq 1$.

Sol. : Assuming $f(k) = 0$ for $k \leq 0$,

$$Z\{f(k)\} = \sum_{k=-\infty}^{0} 0 \cdot z^{-k} + \sum_{k=1}^{\infty} \frac{\alpha^k}{k} z^{-k}$$

$$= \frac{\alpha}{z} + \frac{\alpha^2}{2z^2} + \frac{\alpha^3}{3z^3} + \frac{\alpha^4}{4z^4} + ... \qquad ....(1)$$

$$Z\{f(k)\} = -\log\left(1-\frac{\alpha}{z}\right)$$

Applying D'Alembert's Ratio Test to (1), we **find that the** series is convergent if $|z/\alpha| > 1$.

$\therefore$ ROC is $|z| > |\alpha|$.

Particular cases : (i) Find Z-transform of $f(k) = \dfrac{1}{k}$, $k \ge 1$.

Sol. : Put $\alpha = 1$ in the above example. [**Ans. :** $-\log\left(1-\dfrac{1}{z}\right)$, $|z| > 1$]

(ii) Find the Z-transform of $f(k) = \dfrac{2^k}{k}$, $k \ge 1$.

Sol. : Put $\alpha = 2$ in the above example. [**Ans. :** $-\log\left(1-\dfrac{2}{z}\right)$, $|z| > 2$]

Ex. 6 : Find the Z-transform of $f(k) = a^k$, $k \ge 0$.

Sol. : Assuming that $f(k) = 0$ when $k < 0$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} a^k z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} a^k z^{-k}$$

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \ldots\ldots = \frac{1}{1-(a/z)} = \frac{z}{z-a}$$

The series being G.P. is convergent if $1 > |a/z|$ *i.e.* $|z| > |a|$.

$\therefore$ ROC is $|z| > |a|$.

Since, $Z\{a^k\} = \dfrac{z}{z-a}$, $Z^{-1}\left[\dfrac{z}{z-a}\right] = a^k$, $k \ge 0$.

Ex. 7 : Find the Z-transform of $f(k) = b^k$, $k < 0$.

Sol. : Assuming that $f(k) = 0$ when $k \ge 0$.

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\, z^{-k} = \sum_{k=-\infty}^{-1} b^k z^{-k} = \sum_{n=1}^{\infty} b^{-n} z^n \text{ where } n = -k$$

(Note the substitution $n = -k$)

$$= \frac{z}{b} + \frac{z^2}{b^2} + \frac{z^3}{b^3} + \ldots\ldots = \frac{z}{b}\left(1 + \frac{z}{b} + \frac{z^2}{b^2} + \ldots\ldots\right)$$

$$= \frac{z}{b}\,\frac{1}{1-(z/b)} = \frac{z}{b-z}$$

The series being G.P. is convergent if $1 > |z/b|$ *i.e.* $|b| > |z|$.

$\therefore$ ROC is $|z| < |b|$.

Since, $Z\{b^k\} = \dfrac{z}{b-z}$, $Z^{-1}\left[\dfrac{z}{b-z}\right] = b^k$, $k < 0$.

Ex. 8 : Find Z-transform of $f(k) = \begin{cases} b^k, & k < 0 \\ a^k, & k \ge 0 \end{cases}$

Sol. : By example 7 and 6, we get

$$Z\{f(k)\} = \frac{z}{a-z} + \frac{z}{z-b}$$

$$= \frac{z(a-b)}{(a-z)(z-b)};\quad a > |z| > b$$

[**Note :** In general the Z-transform of $\{f(k)\} = b^k$, for $k < 0$ and $\{f(k)\} = a^k$ for $k \ge 0$ is $\dfrac{(b-a)z}{(b-z)(z-a)}$ and $a < |z| < b$. This Z-transform exists only if $a < b$.]

Ex. 9 : Find the Z-transform of $f(k) = {}^nC_k$, $0 \le k \le n$.

Sol. :
$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {}^nC_k z^{-k} = \sum_{k=0}^{n} {}^nC_k z^{-k}$$

$$= {}^nC_0 + {}^nC_1 \frac{1}{z} + {}^nC_2 \frac{1}{z^2} + \ldots\ldots + {}^nC_n \frac{1}{z^n} = \left(1+\frac{1}{z}\right)^n$$

The series being finite is obviously convergent if $z \ne 0$.

$\therefore$ ROC is all of z-plane except the origin.

Ex. 10 : Find the Z-transform of $f(k) = {}^kC_n$, $k \ge n$.

Sol. :
$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {}^kC_n z^{-k} = \sum_{k=n}^{\infty} {}^kC_n z^{-k}$$

To find the sum we change the dummy index k by $k = n + r$.

$$\therefore\quad Z\{f(k)\} = \sum_{r=0}^{\infty} {}^{n+r}C_n z^{-(n+r)} = \sum_{r=0}^{\infty} {}^{n+r}C_n z^{-n} z^{-r}$$

$$= z^{-n} \sum_{r=0}^{\infty} {}^{n+r}C_r\, z^{-r}$$

[$\because$ ${}^nC_r = {}^nC_{n-r}$, we get ${}^{n+r}C_n = {}^{n+r}C_{n+r-n} = {}^{n+r}C_r$]

$$= z^{-n}\left[1 + {}^{n+1}C_1 z^{-1} + {}^{n+1}C_2 z^{-2} + \ldots\ldots\right] = z^{-n}\left(1-\frac{1}{z}\right)^{-(n+1)}$$

$\therefore$ ROC is $|z| > 1$.

Ex. 11 : Find the Z-transform of $f(k) = {}^{k+n}C_n$.

Sol. :
$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {}^{k+n}C_n z^{-k}$$

But ${}^{k+n}C_n = 0$ if $k + n < n$ *i.e.* if $k < 0$.

$$\therefore \quad Z\{f(k)\} = \sum_{k=0}^{\infty} {}^{k+n}C_n z^{-k} = \sum_{k=0}^{\infty} {}^{k+n}C_k z^{-k}$$

(as in the previous example)

$$= 1 + {}^{n+1}C_1 z^{-1} + {}^{n+2}C_2 z^{-2} + \ldots\ldots$$

$$= \left(1 - \frac{1}{z}\right)^{-(n+1)}$$

As above ROC is $|z| > 1$.

Ex. 12 : Find the Z-transform of $f(k) = \dfrac{a^k}{k!}$, $k \geq 0$.

Sol. :
$$Z\{f(k)\} = \sum_{k=0}^{\infty} \frac{a^k}{k!} \cdot z^{-k} = \sum_{k=0}^{\infty} \frac{(a/z)^k}{k!}$$

$$= 1 + \frac{a}{z} + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \ldots\ldots$$

$$= e^{a/z} \text{ ROC is all of } z\text{-plane.}$$

Remark : You are advised to memorise the Z-transforms of these standard function.

EXERCISE

Find the Z-transform and its ROC of each of the following sequences.

1. $f(k) = 3^k$, $k \geq 0$
2. $f(k) = 4^k$, $k \geq 0$
3. $f(k) = (1/6)^k$, $k \geq 0$
4. $f(k) = 2$, $k \geq 0$
5. $f(k) = 4$, $k \geq 0$
6. $f(k) = 5$, $k \geq 0$
7. $f(k) = 2^k$, $k < 0$
8. $f(k) = 4^k$, $k < 0$
9. $f(k) = (1/3)^k$, $k < 0$
10. $f(k) = 3^k$, $k < 0$; $= 2^k$, $k \geq 0$
11. $f(k) = 4^k$, $k < 0$; $= 3^k$, $k \geq 0$
12. $f(k) = a^k$, $k < 0$; $= b^k$, $k \geq 0$ $(a, b > 0,\ a > b)$
13. $f(k) = k3^k$, $k \geq 0$
14. $f(k) = k\,5^k$, $k \geq 0$
15. $f(k) = ka^k$, $k \geq 0$ $(a > 0)$
16. $f(k) = \dfrac{3^k}{k}$, $k > 1$
17. $f(k) = \dfrac{2^k}{k}$, $k > 1$
18. $f(k) = \dfrac{a^k}{k}$, $k > 1$, $a > 0$
19. $f(k) = (1/2)^{|k|}$, for all k
20. $f(k) = (1/4)^{|k|}$, for all k
21. $f(k) = a^k$ for all k $(0 < a < 1)$
22. $f(k) = (3^k / k!)$, $k \geq 0$
23. $f(k) = (5^k / k!)$, $k \geq 0$
24. $f(k) = e^{k\alpha}$, $k \geq 0$
25. $f(k) = \begin{cases} 2^k, & k \leq -1\ldots.. \\ (1/2)^k, & k = 0, 2, 4\ldots.. \\ (1/3)^k, & k = 1, 3, 5\ldots.. \end{cases}$

[**Ans. : (1)** $\dfrac{1}{1-(3/z)}$; $|z| > 3$, **(2)** $\dfrac{1}{1-(4/z)}$; $|z| > 4$, **(3)** $\dfrac{1}{1-(1/6z)}$; $|z| > \dfrac{1}{6}$

(4) $2 \cdot \dfrac{1}{1-(1/z)}$; $|z| > 1$, **(5)** $4 \cdot \dfrac{1}{1-(1/z)}$; $|z| > 1$, **(6)** $5 \cdot \dfrac{1}{1-(1/z)}$; $|z| > 1$,

(7) $\dfrac{z}{2} \cdot \dfrac{1}{1-(z/2)}$; $|z| < 2$, **(8)** $\dfrac{z}{4} \cdot \dfrac{1}{1-(z/4)}$; $|z| < 4$, **(9)** $3z \dfrac{1}{1-3z}$; $|z| < \dfrac{1}{3}$,

(10) $\dfrac{2z}{(3-z)(z-2)}$; $2 < |z| < 3$, **(11)** $\dfrac{3z}{(4-z)(z-3)}$; $3 < |z| < 4$,

(12) $\dfrac{(a-b)\,z}{(a-z)(z-b)}$; $b < |z| < a$, **(13)** $\dfrac{3z}{(z-3)^2}$; $|z| > 3$,

(14) $\dfrac{5z}{(z-5)^2}$; $|z| > 5$, **(15)** $\dfrac{az}{(z-a)^2}$; $|z| > a$, **(16)** $-\log\left(1 - \dfrac{3}{z}\right)$; $|z| > 3$,

(17) $-\log\left(1 - \dfrac{2}{z}\right)$; $|z| > 2$, **(18)** $-\log\left(1 - \dfrac{a}{z}\right)$; $|z| > a$,

(19) $\dfrac{1}{2} \cdot \dfrac{z}{1-(z/2)} + \dfrac{1}{1-2z}$; $\dfrac{1}{2} < |z| < 2$,

(20) $\dfrac{1}{4} \cdot \dfrac{z}{1-(z/4)} + \dfrac{1}{1-(1/4z)}$; $\dfrac{1}{4} < |z| < 4$,

(21) $\dfrac{az}{1-az} + \dfrac{1}{1-(a/z)}$; $|a| < |z| < \dfrac{1}{|a|}$,

(22) $e^{3/z}$, ROC z plane, **(23)** $e^{5/z}$, ROC z plane,

(24) $\left(1 - \dfrac{e^{\alpha}}{z}\right)^{-1}$; $|z| > |e^{\alpha}|$,

(25) $Z\{f(k)\} = \displaystyle\sum_{-1}^{-\infty} 2^k z^{-k} + \sum_{k=0}^{2n} \left(\frac{1}{2}\right)^k z^{-k} + \sum_{k=0}^{2n-1} \left(\frac{1}{3}\right)^k z^{-k}$ as $n \to \infty$

$$= \sum_{1}^{\infty} \left(\frac{z}{2}\right)^k + \sum_{k=0}^{2n} \left(\frac{1}{2z}\right)^k + \sum_{k=1}^{2n-1} \left(\frac{1}{3z}\right)^k$$

$$= \frac{z}{2-z} + \frac{4z^2}{4z^2-1} + \frac{3z}{9z^2-1} \quad ; \quad \frac{1}{2} < |z| < 2 \;]$$

9. Properties of Z-transforms

As in the Laplace transform we have the following properties of Z-transform. We shall prove below these properties and use them is some problems.

(1) Linearity

If a and b are constants and $\{f(k)\}$ and $\{g(k)\}$ are two sequences which can be added then,

$$Z\{a\,f(k) + b\,g(k)\} = a\,Z\{f(k)\} + b\,Z\{g(k)\}$$

Proof : We have by definition,

$$Z\{af(k) + b\,g(k)\} = \sum_{k=-\infty}^{\infty} [a\,f(k) + b\,g(k)]\,z^{-k}$$

$$= \Sigma[a\,f(k)\,z^{-k} + b\,g(k)\,z^{-k}]$$

$$= a\,\Sigma f(k)\,z^{-k} + b\,\Sigma\,g(k)\,z^{-k}$$

$$= a\,Z\{f(k)\} + b\,Z\{g(k)\}$$

Corollary : If $a = b = 1$, we have

$$Z\{f(k) + g(k)\} = Z\{f(k)\} + Z\{g(k)\}$$

where, $\{f(k)\}$ and $\{g(k)\}$ can be added.

In words, this means, the Z-transform of the sum (or difference when $b = -1$) of two sequences which can be added (or subtracted) is equal to the sum (or difference) of the Z-transforms of the two sequences.

We shall now use this property to solve some problems.

Ex. 1 : Find $Z\{a^{|k|}\}$.

Sol. : We have

$$Z\{a^{|k|}\} = \sum_{k=-\infty}^{\infty} a^{|k|}\,z^{-k}$$

$$= \sum_{k=-\infty}^{-1} a^{-k}\,z^{-k} + \sum_{k=0}^{\infty} a^{k}\,z^{-k}$$

$$= [.....+ a^3z^3 + a^2z^2 + az] + [1 + az^{-1} + a^2x^{-2} + a^3z^{-3} + ...]$$

$$= az\,(1 + az + az^2 + a^3z^3 + ...) + \left(1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + ...\right)$$

$$= az \cdot \frac{1}{1-az} + 1 \cdot \frac{1}{1-(a/z)} = \frac{az}{1-az} + \frac{z}{z-a}$$

$$= \frac{az^2 - a^2z + z - az^2}{(1-az)(z-a)}$$

$$= \frac{z\,(1-a^2)}{(1-az)(z-a)}.$$

The series in G.P. are convergent if $1 > |\,az\,|$ and $1 > \left|\frac{a}{z}\right|$ *i.e.* $\frac{1}{a} > |z|$ and $|\,z\,| > a$.

$\therefore$ The ROC is $(1/a) > |\,z\,| > a$.

Ex. 2 : Find the Z-transform of $\left\{\left(\frac{1}{3}\right)^{|k|}\right\}$.

Sol. : We have

$$Z\left\{\left(\frac{1}{3}\right)^{|k|}\right\} = \Sigma\left(\frac{1}{3}\right)^{|k|} \cdot z^{-k}$$

$$= \sum_{k=-\infty}^{-1}\left(\frac{1}{3}\right)^{-k} z^{-k} + \sum_{k=\infty}^{\infty}\left(\frac{1}{3}\right)^{k} z^{-k}$$

$$= \left[....+\left(\frac{1}{3}\right)^3 z^3 + \left(\frac{1}{3}\right)^2 z^2 + \left(\frac{1}{3}\right) z\right]$$

$$+ \left[1 + \frac{1}{3}\cdot z^{-1} + \left(\frac{1}{3}\right)^2 z^{-2} + \left(\frac{1}{3}\right)^3 z^{-3} + ...\right]$$

$$= \left[\frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + ...\right] + \left[1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \frac{1}{(3z)^3} + ...\right]$$

$$= \frac{z}{3}\left[1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + ...\right]\left[1 + \left(\frac{1}{3z}\right) + \left(\frac{1}{3z}\right)^2 + ...\right]$$

$$= \frac{z}{3} \cdot \frac{1}{1-(z/3)} + 1 \cdot \frac{1}{1-[1/(3z)]}, \quad \left|\frac{z}{3}\right| < 1, \left|\frac{1}{3z}\right| < 1$$

$$= \frac{z}{3} \cdot \frac{3}{3-z} + \frac{3z}{3z-1} = \frac{z}{3-z} + \frac{3z}{3z-1}, \quad |z| < 3, \frac{1}{3} < |z|$$

$$= \frac{3z^2 - z + 9z - 3z^2}{(3-z)(3z-1)} = \frac{8z}{(3-z)(3z-1)}, \quad \frac{1}{3} < |z| < 3.$$

(**Remark :** The above Ex. 2 is a particular case of Ex. 1 where $a = 1/3$.)

Ex. 3 : Find the Z-transform $\{3^{|k|}\}$.

Sol. : Putting $a = 3$ in Ex. 1 or proceeding independently, we get

$$Z\{3^{|k|}\} = \frac{-8z}{(1-3z)(z-3)}$$

Ex. 4 : Find the Z-transform of $f(k) = c^k \cos \alpha k$, $k \geq 0$, where α is real.

Sol. : Assuming $f(k) = 0$ for $k < 0$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\,z^{-k} = \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} c^k \cos\alpha k\,z^{-k}$$

$$= \sum_{k=0}^{\infty} c^k \left[\frac{e^{i\alpha k} + e^{-i\alpha k}}{2}\right] z^{-k}$$

$$= \sum_{k=0}^{\infty} c^k \cdot \frac{e^{i\alpha k}}{2} z^{-k} + \sum_{k=0}^{\infty} c^k \cdot e^{-i\alpha k} z^{-k}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{c\,e^{i\alpha}}{z} \right)^k + \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{c\,e^{-i\alpha}}{z} \right)^k$$

$$= \frac{1}{2} \cdot \left[\frac{1}{1-(ce^{i\alpha}/z)} \right] + \frac{1}{2} \left[\frac{1}{1-(ce^{-i\alpha}/z)} \right] \qquad \text{.......... (1)}$$

[See Note 3 page 1-6]

$$= \frac{1}{2} \left[\frac{z}{z-c\,e^{i\alpha}} + \frac{z}{z-c\,e^{-i\alpha}} \right] = \frac{z}{2} \left[\frac{z-c\,e^{-i\alpha}+z-ce^{i\alpha}}{z^2-zc(e^{i\alpha}+e^{-i\alpha})+c^2} \right]$$

$$= \frac{z}{2} \left[\frac{2z-2c(e^{i\alpha}+e^{-i\alpha})/2}{z^2-2zc(e^{i\alpha}+e^{-i\alpha})/2+c^2} \right]$$

$$= \frac{z(z-c\cos\alpha)}{z^2-2zc\cos\alpha+c^2} \qquad \text{.......... (2)}$$

From (1), we find that the series being G.P. are convergent if $|z|>|ce^{i\alpha}|$ and $|z|>|ce^{-i\alpha}|$ *i.e.* if $|z| > |c(\cos\alpha \pm i\sin\alpha)|$ *i.e.* if $|z| > |c|$.

Ex. 5 : Find the Z-transform of

(i) $f(k) = \cos\alpha k,\ k > 0$ where α is real.

(ii) $f(k) = \cos\dfrac{k\pi}{3}$.

Sol. : Put $c = 1$ in the above example.

$$Z\{f(k)\} = \frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1};\ |z|>1$$

Putting $\alpha = \dfrac{\pi}{3}$, $Z\left\{\cos\dfrac{k\pi}{3}\right\} = \dfrac{z(z-1/2)}{z^2-z+1}$; $|z|>1$

Ex. 6 : Find the Z-transform of $f(k) = c^k \sin\alpha k,\ k \ge 0$.

Sol. : Following the above lines we find that

$$Z\{f(k)\} = \frac{c\,z\sin\alpha}{z^2-2cz\cos\alpha+c^2},\ |z|>|c|$$

Ex. 7 : Find the Z-transform of

(i) $f(k) = \sin\alpha k,\ k \ge 0$ where α is real.

(ii) $f(k) = \sin\dfrac{k\pi}{3}$.

Sol. : Put $c = 1$ in the above example.

$$Z\{f(k)\} = \frac{z\sin\alpha}{z^2-2z\cos\alpha+1};\ |z|>1$$

Putting $\alpha = \dfrac{\pi}{3}$, $Z\left\{\sin\dfrac{k\pi}{3}\right\} = \dfrac{\sqrt{3}\,z/2}{z^2-z+1}$.

Ex. 8 : Find the Z-transform of $f(k) = c^k \cos h\,\alpha k,\ k \ge 0$.

Sol. : $Z\{f(k)\} = \sum\limits_{k=-\infty}^{\infty} f(k)\,z^{-k} = \sum\limits_{k=0}^{\infty} c^k \cos h\,\alpha k\, z^{-k}$

$$= \sum_{k=0}^{\infty} c^k \left(\frac{e^{\alpha k}+e^{-\alpha k}}{2} \right) z^{-k}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{c\,e^{\alpha}}{z} \right)^k + \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{c\,e^{-\alpha}}{z} \right)^k$$

$$= \frac{1}{2} \left[\frac{1}{1-(ce^{\alpha}/z)} \right] + \frac{1}{2} \left[\frac{1}{1-(ce^{-\alpha}/z)} \right] \qquad \text{.......... (1)}$$

[By note (3), page 1-6]

$$= \frac{1}{2} \left[\frac{z}{z-c\,e^{\alpha}} + \frac{z}{z-c\,e^{-\alpha}} \right] = \frac{z}{2} \left[\frac{z-c\,e^{-\alpha}+z-c\,e^{\alpha}}{z^2-zc\,(e^{\alpha}+e^{-\alpha})+c^2} \right]$$

$$= \frac{z}{2} \left[\frac{2z-2c\cos h\alpha}{z^2-2zc\cos h\alpha+c^2} \right] = \frac{z(z-c\cos h\alpha)}{z^2-2zc\cos h\alpha+c^2}$$

From (1), we see that the series are convergent if $|z| > |ce^{\alpha}|$ and $|z| > |ce^{-\alpha}|$ respectively *i.e.* $|z| >$ max. of $|ce^{\alpha}|$ and $|ce^{-\alpha}|$.

Ex. 9 : Find the Z-transform of

$$f(k) = c^k \sin h\,\alpha\, k,\ k \ge 0$$

Sol. : Following the above lines, we find that

$$Z\{f(k)\} = \frac{cz\sin h\,\alpha}{z^2-2z\cos h\alpha+1},\ |z| > \text{max.}|e^{\alpha}| \text{ and } |e^{-\alpha}|$$

Ex. 10 : Find the Z-transform of $\{\sin(ak+b)\},\ k \ge 0$.

Sol. : We have $\sin(ak+b) = \sin ak\cos b + \cos ak\sin b$

$\therefore\ Z\{\sin(ak+b)\} = \cos b \cdot Z\{\sin ak\} + \sin b \cdot Z\{\cos ak\}$

$$= \cos b \cdot \frac{z\sin a}{z^2-2z\cos a+1} + \sin b \cdot \frac{z(z-\cos a)}{z^2-2z\cos a+1}$$

$$= \frac{z[\sin a\cos b - \cos a\sin b + z\sin b]}{z^2-2z\cos a+1}$$

$$= \frac{z[\sin(a-b)+z\sin b]}{z^2-2z\cos a+1}.$$

Ex. 11 : Find the Z-transform of $\{\cos(ak+b)\},\ k \ge 0$.

Sol. : We have $\cos(ak+b) = \cos ak\cos b - \sin ak\sin b$

$\therefore\ Z\{\cos(ak+b)\} = \cos b \cdot Z\{\cos ak\} - \sin b \cdot Z\{\sin ak\}$ ·

$$= \cos b \cdot \frac{z(z-\cos a)}{z^2-2z\cos a+1} - \sin b \cdot \frac{z\sin a}{z^2-2z\cos a+1}$$

$$= \frac{z\,[z\cos b - (\cos a \cos b + \sin a \sin b)]}{z^2 - 2z\cos a + 1}$$

$$= \frac{z\,[z\cos b - \cos(a - b)]}{z^2 - 2z\cos a + 1}.$$

Ex. 12 : Find the Z-transform of $\left\{\cos\left(\frac{k\pi}{3} + \alpha\right)\right\}$, $k \geq 0$.

Sol. : We have

$$Z\{f(k)\} = Z\left\{\cos\left(\frac{k\pi}{3} + \alpha\right)\right\}$$

$$= Z\left\{\cos\frac{k\pi}{3}\cdot\cos\alpha - \sin\frac{k\pi}{3}\cdot\sin\alpha\right\}$$

$$= \cos\alpha \cdot Z\left\{\cos\frac{k\pi}{3}\right\} - \sin\alpha \cdot Z\left\{\sin\frac{k\pi}{3}\right\}$$

$$\therefore \quad F(z) = \cos\alpha \cdot \frac{z^2 - z\cos(\pi/3)\cdot 1}{z^2 - 2z\cdot\cos(\pi/3) + 1} - \sin\alpha \cdot \frac{z\sin(\pi/3)}{z^2 - 2z\cdot\cos(\pi/3) + 1}$$

$$= \frac{z\,\{z\cos\alpha - [\cos(\pi/3)\cdot\cos\alpha + \sin(\pi/3)\sin\alpha]\}}{z^2 - 2\cos(\pi/3)\cdot z + 1}$$

$$= \frac{z\,\{z\cos\alpha - \cos[(\pi/3) - \alpha]\}}{z^2 - 2\cos(\pi/3)\cdot z + 1}$$

EXERCISE

Find the Z-transforms of the following ($k \geq 0$).

1. $2^{|k|}$ 2. $\left(\frac{1}{2}\right)^k$ 3. $\cos k$ 4. $\cos 2k$

5. $\sin k$ 6. $\sin 2k$ 7. $\cos hk$ 8. $\cos h\,2k$

9. $\sin h\,k$ 10. $\sin h\,2k$ 11. $\sin(k+1)$ 12. $2^k \cos k$

13. $\sin(3k+2)$ 14. $\cos(3k+2)$ 15. $\sin\left(\alpha k + \frac{\pi}{2}\right)$ 16. $\cos\left(\alpha k + \frac{\pi}{2}\right)$

[**Ans. :** (1) $-\frac{3z}{(1-2z)(z-2)}$, (2) $\frac{3z}{(2-z)(2z-1)}$, (3) $\frac{z(z-\cos 1)}{z^2 - 2z\cos 1 + 1}$,

(4) $\frac{z(z-\cos 2)}{z^2 - 2z\cos 2 + 1}$, (5) $\frac{z\sin 1}{z^2 - 2\cos 1 + 1}$, (6) $\frac{z\sin 2}{z^2 - 2\cos 2 + 1}$,

(7) $\frac{z(z-\cos h1)}{z^2 - 2z\cos h1 + 1}$, (8) $\frac{z(z-\cos h2)}{z^2 - 2z\cos h2 + 1}$, (9) $\frac{\sin h1}{z^2 - 2z\cos h1 + 1}$,

(10) $\frac{\sin h2}{z^2 - 2z\cos h2 + 1}$, (11) $\frac{z^2 \sin 1}{z^2 - 2z\cos 1 + 1}$,

(12) $\frac{z^2 - 2\cos 1\cdot z}{z^2 - 4\cos 1\cdot z + 4}$, (13) $\frac{z(\sin 1 + z\sin 2)}{z^2 - 2z\cos 3 + 1}$,

(14) $\frac{z(z\cos 2 - \cos 1)}{z^2 - 2z\cos 3 + 1}$, (15) $\frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$,

(16) $\frac{-z\cos\alpha}{z^2 - 2z\cos\alpha + 1}$.].

(2) Change of Scale

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{a^k f(k)\} = F(z/a)$ and if ROC of $Z\{f(k)\}$ is $R_1 < |z| < R_2$, then ROC of $Z\{a^k f(k)\}$ is $|a|\,R_1 < |z| < |a|\,R_2$.

Proof : By definition,

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\,z^{-k}$$

Replacing z by z/a, we get,

$$F\left(\frac{z}{a}\right) = \Sigma f(k)\left(\frac{z}{a}\right)^{-k} \qquad \text{.......... (1)}$$

But by definition again,

$$Z\{a^k f(k)\} = \sum_{k=-\infty}^{\infty} a^k f(k)\,z^{-k} = \Sigma f(z)\left(\frac{z}{a}\right)^{-k} \qquad \text{.......... (2)}$$

From (1) and (2), we get

$$Z\left\{a^k f(k)\right\} = F\left(\frac{z}{a}\right)$$

Further, if ROC of $Z\{f(k)\}$ is $R_1 < |z| < R_2$, then ROC of $Z\{a^k f(k)\}$ from (2) will be $R_1 < |z/a| < R_2$ *i.e.* $|a|\,R_1 < |z| < |a|\,R_2$.

Ex. 1 : Obtain $Z\{1\}$ and hence deduce $Z\{a^k\}$, $k \geq 0$.

Sol. : By definition

$$Z\{1\} = \sum_{k=0}^{\infty} 1\cdot z^{-k}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots..$$

$$= \frac{1}{1-(1/z)} = \frac{z}{z-1}.$$

Now, $a^k = a^k \cdot 1$. Hence, by change of scale property,

$$Z\{a^k\} = Z\{a^k \cdot 1\} = \frac{z/a}{(z/a) - 1} = \frac{z}{z-a}.$$

Ex. 2 : Find $Z\{c^k \sin\alpha k\}$ from $Z\{\sin\alpha k\}$

Sol. : We know that,

$$Z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$$

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By change of scale property,

$$Z\{c^k \sin\alpha k\} = \frac{(z/c)\sin\alpha}{(z/c)^2 - 2(z/c)\cos\alpha + 1} = \frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2}.$$

Ex. 3 : Find $Z\{c^k \cos\alpha k\}$ from $Z\{\cos\alpha k\}$.

Sol. : We know that

$$Z\{\cos\alpha k\} = \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1}$$

By change of scale property,

$$Z\{c^k \cos\alpha k\} = \frac{\frac{z}{c}\left\{\frac{z}{c} - \cos\alpha\right\}}{\left(\frac{z}{c}\right)^2 - 2\left(\frac{z}{c}\right)\cos\alpha + 1} = \frac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2}.$$

(3) Shifting Property

Theorem : If $Z\{f(k)\} = F(z)$, then

$$Z\{f(k+n)\} = z^n F(z) \text{ and } Z\{f(k-n)\} = z^{-n} F(z)$$

Proof : We have

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = F(z)$$

$$\therefore \quad Z\{f(k+n)\} = \Sigma f(k+n) z^{-k} = \Sigma f(k+n) z^{-(k+n)} \cdot z^n = z^n \sum_{k=-\infty}^{\infty} f(k+n) z^{-(k+n)}$$

If we put $k + n = m$ when $k = -\infty$, $m = -\infty$ and when $k = +\infty$, $m = +\infty$.

$$\therefore \quad Z\{f(k+n)\} = z^n \sum_{m=-\infty}^{\infty} f(m) z^{-m} = z^n F(z)$$

Changing the sign of n or proceeding as above.

$$Z\{f(k-n)\} = z^{-n} F(z).$$

(a) Unilateral or one sided or casual sequence : If a sequence $\{f(k)\}$ is defined for the right side only *i.e.* if $\{f(k)\}$ extends to infinity on the right only *i.e.* if k varies from 0 to $+\infty$, the sequence is called **unilateral** or **one sided** or **casual sequence**.

For example, $f(k) = \begin{cases} 0, & k < 0 \\ 2^k, & k \geq 0 \end{cases}$ is a causal sequences.

Theorem : If $\{f(k)\}$ is one sided and if

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z),$$

then $$Z\{f(k+n)\} = z^n F(z) - \sum_{m=0}^{n-1} f(m) z^{n-m}$$

and $$Z\{f(k-n)\} = z^{-n} F(z) + \sum_{r=1}^{n} f(-r) z^{-n+r}$$

Proof : Since $\{f(k)\}$ is one sided sequence, by data,

$$F(z) = Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$\therefore \quad Z\{f(k+n)\} = \sum_{k=0}^{\infty} f(k+n) z^{-k} = \sum_{k=0}^{\infty} f(k+n) z^{-(k+n)} \cdot z^n = z^n \sum_{k=0}^{\infty} f(k+n) z^{-(k+n)}$$

Put $k + n = m$. When $k = 0$, $n = m$; when $k = \infty$, $m = \infty$.

$$\therefore \quad Z\{f(k+n)\} = z^n \sum_{m=n}^{\infty} f(m) z^{-m}$$

The interval from n to ∞ can be changed to the interval from 0 to ∞ by subtracting from it, the interval 0 to $n - 1$.

$$\therefore \quad Z\{f(k+n)\} = z^n \sum_{m=0}^{\infty} f(m) z^{-m} - z^n \sum_{m=0}^{n-1} f(m) z^{-m} = z^n \cdot F(z) - z^n \sum_{m=0}^{n-1} f(m) z^{-m}$$

Taking z^n inside the summation,

$$\therefore \quad Z\{f(k+n)\} = z^n F(z) - \sum_{m=0}^{n-1} f(m) z^{n-m}.$$

Further, $$Z\{f(k-n)\} = \sum_{k=0}^{\infty} f(k-n) z^{-k} = \sum_{k=0}^{\infty} f(k-n) z^{-(k-n)} \cdot z^{-n} = z^{-n} \sum_{k=0}^{\infty} f(k-n) z^{-(k-n)}$$

Put $k - n = m$. When $k = 0$, $m = -n$ and when $k = \infty$, $m = \infty$.

$$\therefore \quad Z\{f(k-n)\} = z^{-n} \sum_{k=-n}^{\infty} f(m) z^{-m}$$

As before we split the interval from $-n$ to ∞ into two intervals, $-n$ to -1 and 0 to ∞.

$$\therefore \quad Z\{f(k-n)\} = z^{-n}\sum_{m=-n}^{-1} f(m)\,z^{-m} + z^{-n}\sum_{m=0}^{\infty} f(m)\,z^{-m}$$

$$= z^{-n}F(z) + z^{-n}\sum_{m=-n}^{-1} f(m)\,z^{-m}$$

Taking z^{-n} inside the summation,

$$\therefore \quad Z\{f(k-n)\} = z^{-n}F(z) + \sum_{m=-n}^{-1} f(m)\,z^{-(m+n)}$$

Putting $m=-r$, when $m=-1$, $r=1$ and when $m=-n$, $r=n$.

$$\therefore \quad Z\{f(k-n)\} = z^{-n}F(z) + \sum_{r=1}^{n} f(-r)\,z^{-n+r}.$$

Corollary 1 : Since for one sided sequence ($k \geq 0$), we have

$$Z\{f(k-n)\} = z^{-n}F(z) + \sum_{r=1}^{n} f(-r)\,z^{-n+r}$$

Since for causal sequence $f(-1) = f(-2) = \ldots\ldots f(-n) = 0$, the second term is zero.

$$\therefore \quad Z\{f(k-n)\} = z^{-n}F(z).$$

Corollary 2 : Since for one sided sequence ($k \geq 0$), we have

$$Z\{f(k+n)\} = z^{n}F(z) - \sum_{m=0}^{n-1} f(m)\,z^{n-m}$$

Putting $n=1$,

$$Z\{f(k+1)\} = z^{1}F(z) - \sum_{m=0}^{0} f(0)\,z^{1-0}$$

$$= z\,F(z) - z\,f(0)$$

Putting $n=2$,

$$Z\{f(k+2)\} = z^{2}F(z) - \sum_{m=0}^{2-1} f(m)\,z^{2-m}$$

$$= z^2F(z) - \left[f(0)\,z^{2-0} + f(1)\,z^{2-1}\right]$$

$$= z^2F(z) - z^2f(0) - z\,f(1).$$

(4) Multiplication by k

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{k\,f(k)\} = -z\dfrac{d}{dz}F(z)$.

Proof : We have, by definition,

$$Z\{k\,f(k)\} = \sum_{k=-\infty}^{\infty} k\,f(k)\,z^{-k} = \sum k\,f(k)\,z^{-k-1}\cdot z$$

$$= -z\sum -k\,f(k)\,z^{-k-1} = -z\sum f(k)\frac{d}{dz}(z^{-k})$$

$$= -z\cdot\frac{d}{dz}\sum f(k)\,z^{-k} = -z\frac{d}{dz}F(z)$$

In general, $Z\{k^nf(k)\} = \left(-z\dfrac{d}{dz}\right)^n F(z)$

Note that $\left(-z\dfrac{d}{dz}\right)^2 \neq z^2\dfrac{d^2}{dz^2}$, but it is equal to repeated operations $\left(-z\dfrac{d}{dz}\right)\left(-z\dfrac{d}{dz}\right)$.

Corollary 1 : $Z\{k\} = \dfrac{z}{(z-1)^2}$, $|z| > 1$

Proof : By definition,

$$Z\{1\} = \sum_{k=0}^{\infty} 1\cdot z^{-k} = 1 + z^{-1} + z^{-2} + \ldots\ldots$$

$$= \frac{1}{1-z^{-1}} = \frac{1}{1-(1/z)}, \quad |z^{-1}| < 1$$

$$= \frac{z}{z-1}, \quad |z| > 1$$

Now, by the above theorem,

$$Z\{k\} = Z\{k\cdot 1\} = \left(-z\frac{d}{dz}\right)[Z\{1\}]$$

$$= \left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right) = -z\left[\frac{(z-1)1 - z\cdot 1}{(z-1)^2}\right]$$

$$= -z\left[\frac{-1}{(z-1)^2}\right] = \frac{z}{(z-1)^2}, \quad |z| > 1.$$

Corollary 2 : $Z\{k^2\} = \dfrac{z(z+1)}{(z-1)^3}$.

Proof : We have proved above that $Z\{1\} = \dfrac{z}{z-1}$.

By the above theorem,

$$Z\{k^2\} = Z\{k^2\cdot 1\} = \left(-z\frac{d}{dz}\right)^2 Z\{1\}$$

$$= \left(-z\frac{d}{dz}\right)\left(-z\frac{d}{dz}\right)\left(\frac{z}{z-1}\right)$$

$$= \left(-z\frac{d}{dz}\right)\left[\frac{z}{(z-1)^2}\right] \qquad \text{[As above]}$$

$$= -z\left[\frac{(z-1)^2\cdot 1 - z\cdot 2(z-1)\cdot 1}{(z-1)^4}\right]$$

$$= -z\left[\frac{(z-1)-2z}{(z-1)^3}\right] = \frac{z(z+1)}{(z-1)^3}.$$

(5) Initial Value

Theorem : If $Z\{f(k)\} = F(z)$, $k \geq 0$, then $f(0) = \lim_{z\to\infty} F(z)$.

Proof : By definition

$$Z\{f(k)\} = \sum_{k=0}^{\infty} f(k)\, z^{-k} = F(z)$$

$$\therefore \quad f(0)z^0 + f(1)z^{-1} + f(2)z^{-2} + \ldots = F(z)$$

$\therefore$ Taking the limit as $z \to \infty$ of both sides of

$$f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \ldots\ldots = F(z)$$

$$\therefore \quad f(0) = \lim_{z\to\infty} F(z).$$

(6) Final Value

Theorem : $\lim_{k\to 0} f(k) = \lim_{z\to 1} (z-1)F(z)$

Proof : By definition,

$$Z\{f(k+1) - f(k)\} = \sum_{k=0}^{\infty} [f(k+1) - f(k)]\, z^{-k}$$

$$\therefore \quad Z\{f(k+1)\} - Z\{f(k)\} = \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1) - f(k)]\, z^{-k}$$

$$\therefore \quad zF(z) - zf(0) - F(z) = \lim_{n\to\infty} \sum_{k=0}^{n} [f(k+1) - f(k)]\, z^{-k}$$

[By corollary (2), page 1-24]

$$\therefore \quad (z-1)F(z) = zf(0) + \lim_{n\to\infty} \sum [f(k+1) - f(k)]\, z^{-k}$$

$$\therefore \quad \lim_{z\to 1} (z-1)F(z) = f(0) + \lim_{z\to 1} \lim_{n\to\infty} \sum [f(k+1) - f(k)]\, z^{-k}$$

Changing the order of limits,

$$\lim_{z\to 1} (z-1)F(z) = f(0) + \lim_{n\to\infty} \cdot \lim_{z\to 1} \sum_{k=0}^{n} [f(k+1) - f(k)]\cdot z^{-k}$$

$$= f(0) + \lim_{n\to\infty} \sum [f(k+1) - f(k)] \qquad \left[\because \lim_{z\to 1} z^{-k} = 1\right]$$

$$= \lim_{n\to\infty} [f(0) + f(1) - f(0) + f(2) - f(1) + \ldots\ldots + f(n+1) - f(n)]$$

$$= \lim_{n\to\infty} f(n+1) = \lim_{n\to\infty} f(n) = \lim_{k\to\infty} f(k).$$

(7) Convolution

If $\{f(k)\}$ and $\{g(k)\}$ are two sequences then their convolution $\{f(k)\} * \{g(k)\}$ is defined by $h\{k\} = \{f(k)\} * \{g(k)\}$

$$\text{where, } \{h(k)\} = \sum_{m=-\infty}^{\infty} f(m)\, g(k-m)$$

$$= \sum_{m=-\infty}^{\infty} g(m) f(k-m)$$

$$= \{g(k)\} * \{h(k)\}$$

Theorem : If $\{h(k)\}$ is the convolution of two sequences $\{f(k)\}$ and $\{g(k)\}$ then $Z\{h(k)\} = Z\{f(k)\}\, Z\{g(k)\}$ *i.e.* $H(z) = F(z)\, G(z)$.

Proof : By definition,

$$H(z) = Z\{h(k)\} = Z\big[\{f(k)\} * \{g(k)\}\big]$$

$$= Z\left[\sum_{m=-\infty}^{\infty} f(m)\, g(k-m)\right]$$

$$= \sum_{k=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} f(m)\, g(k-m)\right] z^{-k}.$$

Since the power series coverges absolutely, it converges uniformly also within ROC. Hence, we can interchange the order of summation.

$$H(z) = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(m)\; g(k-m)\, z^{-k}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(m)\cdot g(k-m)\, z^{-k+m-m}$$

$$= \sum_{m=-\infty}^{\infty} f(m)\, z^{-m} \cdot \sum_{k=-\infty}^{\infty} g(k-m)\, z^{-(k-m)}$$

$$= \sum_{m=-\infty}^{\infty} f(m)\, z^{-m} \cdot \sum_{p=-\infty}^{\infty} g(p)\, z^{-p} \qquad \text{where, } p = k-m.$$

(When $k = -\infty$, $p = -\infty$ and when $k = \infty$, $p = \infty$)

$$\therefore \quad H(z) = F(z)\; G(z).$$

Ex. 1 : If $f(k) = U(k)$ and $g(k) = 2^k\, U(k)$, find Z-transform of $f(k) * g(k)$.

Sol. : We know that $\{f(k)\} = U(k) = \{1, 1, 1, 1, \ldots\}$

$$\therefore \quad Z\{f(k)\} = \sum 1\cdot z^{-k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\ldots$$

$$\therefore \quad F(z) = \frac{1}{1-(1/z)} = \frac{z}{z-1}, \quad \left|\frac{1}{z}\right| < 1.$$

By the change of scale property,

$$Z\{g(k)\} = Z\{2^k U(k)\} = \frac{z/2}{(z/2)-1}$$

$$G(z) = \frac{z}{z-2}, \quad \left|\frac{2}{z}\right| < 1$$

By convolution theorem,

$$Z\{f(k) * g(k)\} = F(z)\, G(z) = \frac{z}{z-1} \cdot \frac{z}{z-2}$$

$$= \frac{z^2}{(z-1)(z-2)}, \quad |z| > 2.$$

Ex. 2 : If $f(k) = 4^k U(k)$ and $g(k) = 5^k U(k)$, then find the Z-transform of $f(k) * g(k)$.

Sol. : As above, $\{f(k)\} = \{4^0, 4^1, 4^2, ...\}$

$$\therefore \quad Z\{f(k)\} = \sum f(k) z^{-k} = 4^0 z^0 + 4z^{-1} + 4^2 z^{-2} + ...$$

$$= 1 + \frac{4}{z} + \left(\frac{4}{z}\right)^2 + \left(\frac{4}{z}\right)^3 + ...$$

$$= \frac{1}{1-(4/z)} = \frac{z}{z-4}, \quad \left|\frac{4}{z}\right| < 1$$

$$\therefore \quad \{g(k)\} = \{5^0, 5^1, 5^2, ...\}$$

$$\therefore \quad Z\{g(k)\} = \sum g(k) z^{-k} = 5^0 z^0 + 5z^{-1} + 5^2 z^{-2} + ...$$

$$= 1 + \left(\frac{5}{z}\right) + \left(\frac{5}{z}\right)^2 + ...$$

$$= \frac{1}{1-(5/z)} = \frac{z}{z-5}, \quad \left|\frac{5}{z}\right| < 1.$$

$\therefore$ By convolution theorem

$$Z\{f(k) * g(k)\} = F(z) \cdot G(z)$$

$$= \frac{z}{(z-4)} \cdot \frac{z}{(z-5)} = \frac{z^2}{(z-4)(z-5)}.$$

Ex. 3 : Find $Z\{f(k)\}$ where $f(k) = \frac{1}{2^k} * \frac{1}{3^k}$.

Sol. :
$$Z\left\{\frac{1}{2^k}\right\} = \sum_{k=0}^{\infty} \frac{1}{2^k} z^{-k} = 1 + \frac{1}{2^2} + \frac{1}{(2z)^2} + ...$$

$$= \frac{1}{1-[1/(2z)]} = \frac{2z}{2z-1}, \quad |2z| > 1 \ \ i.e.\ \ |z| > \frac{1}{2}.$$

Similarly, $$Z\left\{\frac{1}{3^k}\right\} = \frac{3z}{3z-1}, \quad |z| > \frac{1}{3}.$$

By convolution theorem,

$$Z\{f(k)\} = \left(\frac{2z}{2z-1}\right)\left(\frac{3z}{3z-1}\right), \quad |z| > \frac{1}{2}.$$

Theorem : If $Z\{f(k)\} = F(z)$, then $Z\{e^{-ak} f(k)\} = F(e^a z)$.

Proof : By definition,

$$F(z) = Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$\therefore \quad Z\{e^{-ak} f(k)\} = \sum e^{-ak} f(k) z^{-k}$$

$$= \sum f(k) (e^a z)^{-k} = F(e^a z).$$

Ex. 1 : Find the Z-transform of $\{k e^{-ak}\}$, $k \geq 0$.

Sol. : We know that if $U(k) = 1$, for $k > 0$, then $Z\{U(k)\} = \frac{z}{z-1}$.

By the above theorem, $$Z[e^{-ak} U(k)] = \frac{e^a z}{e^a z - 1}.$$

Now, by (4), page 1-24

$$Z[k e^{-ak}] = -z \frac{d}{dz}\left(\frac{e^a z}{e^a z - 1}\right)$$

$$= -z e^a \cdot \left[\frac{(e^a \cdot z - 1)1 - z(e^a)}{(e^a \cdot z - 1)^2}\right]$$

$$= -z \cdot e^a \cdot \frac{(-1)}{(e^a \cdot z - 1)^2} = \frac{e^a \cdot z}{(e^a z - 1)^2}.$$

Ex. 2 : Find $Z\{e^{-ak} \cos bk\}$.

Sol. : We have already obtained

$$Z\{\cos bk\} = \frac{z(z - \cos b)}{z^2 - 2z\cos b + 1}$$

Now, by the above result,

$$Z\{e^{-ak} \cos bk\} = \frac{e^a z (e^a z - \cos b)}{(e^a \cdot z)^2 - 2(e^a z)\cos b + 1}$$

Multiply in the numerator and denominator by e^{-2a}.

$$Z\{e^{-ak} \cos bk\} = \frac{z(z - e^{-a} \cos b)}{z^2 - 2e^{-a} z \cos b + e^{-2a}}.$$

Ex. 4 : Find $Z\{e^{-ak} \sin bk\}$.

Sol. : We have already proved that

$$Z\{\sin bk\} = \frac{z \sin b}{z^2 - 2z\cos b + 1}$$

$\therefore$ By the above property,

$$Z\{e^{-ak}\sin bk\} = \frac{(e^a z)\sin b}{(e^a z)^2 - 2(e^a z)\cos b + 1}$$

$$= \frac{e^{-a}z \cdot \sin b}{z^2 - 2e^{-a} \cdot z\cos b + e^{-2a}}.$$

Table of Z-transforms

1. $Z[\delta(k)] = 1,$ for all z
2. $Z\{U(k)\} = \dfrac{z}{z-1},$ $|z| > 1$
3. $Z\{1\} = \dfrac{z}{z-1},$ $|z| > 1$
4. $Z\{k\} = \dfrac{z}{(z-1)^2},$ $|z| > 1$
5. $Z\{a^k\} = \dfrac{z}{z-a},$ $k \geq 0,\ |z| > |a|.$
6. $Z\{a^k\} = \dfrac{z}{a-z},$ $k < 0,\ |z| < |a|.$
7. $Z\{k\,a^k\} = \dfrac{az}{(z-a)^2},$ $k \geq 0,\ |z| > |a|.$
8. $Z\{{}^nC_k\} = \left(1 + \dfrac{1}{z}\right)^n,$ $0 \leq k \leq n,\ |z| > 0$
9. $Z\{{}^kC_n\} = z^{-n}\left(1 - \dfrac{1}{z}\right)^{-(n+1)},$ $|z| > 1$
10. $Z\left\{\dfrac{a^k}{k!}\right\} = e^{a/z},$ $k \geq 0$ for all z.
11. $Z\{a^{|k|}\} = \dfrac{az}{1-az} + \dfrac{z}{z-a},$ $|a| < |z| < \dfrac{1}{|a|}$
12. $Z\{c^k\cos\alpha k\} = \dfrac{z(z - c\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2},$ $|z| > |c|$
13. $Z\{c^k\sin\alpha k\} = \dfrac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2},$ $|z| > |c|.$
14. $Z\{c^k\cosh\alpha k\} = \dfrac{z(z - c\cosh\alpha)}{z^2 - 2cz\cosh\alpha + c^2},$ $k \geq 0$

    $|z| > \max(|ce^{\alpha}|, |ce^{-\alpha}|)$
15. $Z\{c^k\sinh\alpha k\} = \dfrac{cz\sinh\alpha}{z^2 - 2cz\cosh\alpha + c^2},$ $k \geq 0$

    $|z| > \max(|ce^{\alpha}|, |ce^{-\alpha}|).$

Miscellaneous Examples

Ex. 1 : Find Z-transform of $\{a\cos k\alpha + b\sin k\alpha\},\ k \geq 0.$

Sol. : $Z\{a\cos k\alpha + b\sin k\alpha\}$

$$= aZ\{\cos k\alpha\} + b\,Z\{\sin k\,a\} \qquad \text{[By linearity property]}$$

$$= a \cdot \frac{z(z-\cos\alpha)}{z^2 - 2z\cos\alpha + 1} + b \cdot \frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}, \quad |z| > 1$$

$$= \frac{az^2 + z(b\sin\alpha - a\cos\alpha)}{z^2 - 2z\cos\alpha + 1}.$$

Ex. 2 : Find $Z\left\{\sin\left(\dfrac{k\pi}{4} + a\right)\right\},\ k \geq 0.$

Sol. : Since $\sin\left(\dfrac{k\pi}{4} + a\right) = \sin\dfrac{k\pi}{4}\cos a + \cos\dfrac{k\pi}{4}\sin a$

$$Z\left\{\sin\left(\frac{k\pi}{4} + a\right)\right\} = Z\left\{\sin\frac{k\pi}{4}\cos a + \cos\frac{k\pi}{4}\sin a\right\}$$

$$= \cos a\,Z \cdot \left\{\sin\left(\frac{\pi}{4}k\right)\right\} + \sin a\,Z \cdot \left\{\cos\left(\frac{\pi}{4} \cdot k\right)\right\}$$

$$= \cos a \frac{z\sin(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1} + \sin a \frac{z[z - \cos(\pi/4)]}{z^2 - 2z\cos(\pi/4) + 1}$$

$$= \frac{\cos a \cdot (z/\sqrt{2})}{z^2 - (2z/\sqrt{2}) + 1} + \frac{\sin a\,z \cdot [z - (1/\sqrt{2})]}{z^2 - (2z/\sqrt{2}) + 1}$$

$$= \frac{z}{\sqrt{2}} \cdot \frac{[\cos a + \sin a \cdot (\sqrt{2}\,z - 1)]}{z^2 - \sqrt{2} \cdot z + 1}.$$

Ex. 3 : Find $Z\{f(k)\}$ where $f(k) = \cos\left(\dfrac{k\pi}{4} + a\right),\ k \geq 0.$

Sol. : We have $\cos\left(\dfrac{k\pi}{4} + a\right) = \cos\dfrac{k\pi}{4}\cos a - \sin\dfrac{k\pi}{4}\sin a$

$$\therefore \quad Z\left\{\cos\left(\frac{k\pi}{4} + a\right)\right\} = Z\left\{\cos\frac{k\pi}{4}\cos a - \sin\frac{k\pi}{4}\sin a\right\}$$

$$= \cos a \cdot Z\left\{\cos\frac{k\pi}{4}\right\} - \sin a \cdot Z\left\{\sin\frac{k\pi}{4}\right\}$$

$$= \cos a \cdot \frac{z[z - \cos(\pi/4)]}{z^2 - 2z\cos(\pi/4) + 1} + \frac{\sin a \cdot z\sin(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1}$$

$$= \frac{\cos a \cdot z[z - (1/\sqrt{2})]}{z^2 - \sqrt{2} \cdot z + 1} - \frac{\sin a \cdot z \cdot 1\sqrt{2}}{z^2 - \sqrt{2} \cdot z + 1}$$

$$= \frac{z}{\sqrt{2}} \cdot \left[\frac{\cos a(\sqrt{2}z - 1) - \sin a}{z^2 - \sqrt{2} \cdot z + 1}\right].$$

Ex. 4 : Find $Z\{2^k \cos(3k+2)\},\ k \geq 0$.

Sol. : We have $\cos(3k+2) = \cos 3k \cos 2 - \sin 3k \sin 2$

$$\therefore\ Z\{\cos(3k+2)\} = \cos 2 \cdot Z\{\cos 3k\} - \sin 2 \cdot Z\{\sin 3k\}$$

$$= \cos 2 \cdot \frac{z(z-\cos 3)}{z^2 - 2z\cos 3 + 1} - \frac{\sin 2 \cdot z \sin 3}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[z\cos 2 - (\cos 3\cos 2 - \sin 3 \sin 2)]}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z \cdot [z\cos 2 - \cos 1]}{z^2 - 2z\cos 3 + 1}$$

By change of scale property,

If $Z\{f(k)\} = F(z)$, then $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$.

$$\therefore\ Z\{2^k \cos(3k+2)\} = \frac{\frac{z}{2}\left[\frac{z}{2}\cdot\cos 2 - \cos 1\right]}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3 + 1}$$

$$= \frac{z[z\cos 2 - 2\cos 1]}{z^2 - 4z\cos 3 + 4}.$$

Ex. 5 : Find $Z\{2^k \sin(3k+2)\},\ k \geq 0$.

Sol. : We have $\sin(3k+2) = \sin 3k \cos 2 + \cos 3k \sin 2$

$$\therefore\ Z\{\sin(3k+2)\} = \cos 2 \cdot Z\{\sin 3k\} + \sin 2 \cdot Z\{\cos 3k\}$$

$$= \cos 2 \cdot \frac{z\sin 3}{z^2 - 2z\cos 3 + 1} + \frac{\sin 2 \cdot z(z-\cos 3)}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[\sin 3\cos 2 - \cos 3 \sin 2 + z\sin 2]}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[\sin(3-2) + z\sin 2]}{z^2 - 2z\cos 3 + 1} = \frac{z[\sin 1 + z\sin 2]}{z^2 - 2z\cos 3 + 1}.$$

Now, by change of scale property as above,

$$\therefore\ Z\{2^k \sin(3k+2)\} = \frac{\frac{z}{2}\left[\sin 1 + \frac{z}{2}\sin 2\right]}{\left(\frac{z}{2}\right)^2 - 2\left(\frac{z}{2}\right)\cos 3 + 1}$$

$$= \frac{z[2\sin 1 + z\sin 2]}{z^2 - 4z\cos 3 + 4}.$$

Ex. 6 : Find $Z\{3^k \sin h\,\alpha\, k\},\ k \geq 0$.

Sol. : We have $Z\{\sin h\,\alpha\, k\} = \dfrac{z \sin h\,\alpha}{z^2 - 2z\cos h\,\alpha + 1}$

By change of scale property,

$$Z\{3^k \sin h\,\alpha\} = \frac{\frac{z}{3}\sin h\,\alpha}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right)\cos h\,\alpha + 1}$$

$$= \frac{3z\sin h\,\alpha}{z^2 - 6z\cos h\,\alpha + 9}.$$

Ex. 7 : Find $Z\{3^k \cos h\,\alpha\, k\},\ k \geq 0$.

Sol. : We have $Z\{\cos h\,\alpha\, k\} = \dfrac{z(z-\cos h\,\alpha)}{z^2 - 2z\cos h\,\alpha + 1}$

By change of scale property,

$$\therefore\ Z\{3^k \cos h\,\alpha\, k\} = \frac{\frac{z}{3}\left(\frac{z}{3} - \cos h\,\alpha\right)}{\left(\frac{z}{3}\right)^2 - 2\left(\frac{z}{3}\right)\cos h\,\alpha + 1}$$

$$= \frac{z(z - 3\cos h\,\alpha)}{z^2 - 6z\cos h\,\alpha + 9}.$$

Ex. 8 : Find $Z\{(k+1)\,a^k\},\ k \geq 0$.

Sol. : We have $Z\{(k+1)\,a^k\} = Z\{k\,a^k\} + Z\{a^k\}$

But we know that $Z\{a^k\} = \dfrac{z}{z-a}$.

Now, by the property of multiplication of k,

$$\therefore\ Z\{k\,a^k\} = -z\frac{d}{dz}\left\{\frac{z}{z-a}\right\} = -z\left[\frac{(z-a)1 - z\cdot 1}{(z-a)^2}\right]$$

$$= -z \cdot \frac{(-a)}{(z-a)^2} = \frac{az}{(z-a)^2}$$

$$\therefore\ Z\{(k+1)\,a^k\} = \frac{az}{(z-a)^2} + \frac{z}{(z-a)}$$

$$= \frac{az + z^2 - az}{(z-a)^2} = \frac{z^2}{(z-a)^2}.$$

Ex. 9 : Find $Z\{k^2 e^{-ak}\},\ k \geq 0$.

Sol. : We have already obtained that, $Z\{e^{-ak}\} = \dfrac{z}{z-e^{-a}}$ [Ex. 1 page 1-29]

$$\therefore\ Z\{k\,e^{-ak}\} = -z\frac{d}{dz}\left(\frac{z}{z-e^{-a}}\right)$$

$$= -z\left[\frac{(z-e^{-a})\cdot 1 - (z)\cdot 1}{(z-e^{-a})^2}\right] = \frac{z\,e^{-a}}{(z-e^{-a})^2}.$$

$$\therefore \quad Z\{k^2 \cdot e^{-ak}\} = Z\left\{k\,(k\,e^{-ak})\right\}$$

$$= -z\frac{d}{dz}\left[\frac{z \cdot e^{-a}}{(z-e^{-a})^2}\right] = -z \cdot e^{-a} \cdot \frac{d}{dz}\left[\frac{z}{(z-e^{-a})^2}\right]$$

$$= -z \cdot e^{-a}\left[\frac{(z-e^{-a})^2 \cdot 1 - z \cdot 2(z-e^{-a}) \cdot 1}{(z-e^{-a})^4}\right]$$

$$= -z \cdot e^{-a}\left[\frac{(z-e^{-a})-2z}{(z-e^{-a})^3}\right]$$

$$= \frac{z\,e^{-a}\,(z+e^{-a})}{(z-e^{-a})^3}, \quad |z| > |e^{-a}|.$$

Ex. 10 : Find Z-transform of $\{k^2 a^{k-1}\}, \ k \geq 0$.

Sol. : We know that, $Z\{f(k-n)\} = z^{-n} \cdot F(z)$

$$\therefore \quad Z\{a^{k-1}\} = z^{-1}F(z) \quad \text{where} \quad F(z) = Z\{a^k\} = \frac{z}{z-a}$$

$$\therefore \quad Z\{a^{k-1}\} = z^{-1} \cdot \frac{z}{z-a} = \frac{1}{z-a}$$

$$\therefore \quad Z\{k \cdot a^{k-1}\} = -z\frac{d}{dz} \cdot \left(\frac{1}{z-a}\right)$$

$$= -z \cdot \frac{(-1)}{(z-a)^2} = \frac{z}{(z-a)^2}.$$

$$\therefore \quad Z\{k^2 \cdot a^{k-1}\} = Z\{k \cdot (ka^{k-1})\}$$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-a)^2}\right] = -z\left[\frac{(z-a)^2 \cdot 1 - z \cdot 2(z-a) \cdot 1}{(z-a)^4}\right]$$

$$= -z\left[\frac{(z-a)-2z}{(z-a)^3}\right] = \frac{z(z+a)}{(z-a)^3}, \quad |z| > |a|.$$

Ex. 11 : Find $Z\{k^2 a^{k-1}\,U(k-1)\}$.

Sol. : We know that, $Z\{U(k)\} = \dfrac{z}{z-1}$.

By change of scale property,

$$Z\{a^k\,U(k)\} = \frac{z/a}{(z/a)-1} = \frac{z}{z-a}$$

$$\because \quad Z\{f(k-n)\} = z^{-n} \cdot Z\{f(k)\}$$

$$Z\{a^k\,U(k-1)\} = z^{-1} \cdot Z\{a^k\,U(k)\}$$

$$= z^{-1} \cdot \frac{z}{z-a} = \frac{1}{z-a}.$$

$$\therefore \quad Z\{k^2 \cdot a^2\,U(k-1)\} = \left(-z\frac{d}{dz}\right)^2\left(\frac{1}{z-a}\right)$$

$$= \frac{z(z+a)}{(z-a)^3} \text{ as above.}$$

EXERCISE

Find the Z-transforms of the following.

1. $\{3^k + 5^k\}, \ k < 0$
2. $\{\alpha^k + \beta^k\}, \ k < 0$
3. $\{3^k + 5^k\}, \ k \geq 0$
4. $\{\alpha^k + \beta^k\}, \ k \geq 0$
5. $\{k2^k + k3^k\}, \ k \geq 0$
6. $\{k\alpha^k + k\beta^k\}, \ k \geq 0$
7. $\left\{\dfrac{2^k}{k} + \dfrac{3^k}{k}\right\}, \ k \geq 0$
8. $\left\{\dfrac{\alpha^k}{k} + \dfrac{\beta^k}{k}\right\}, \ k \geq 0$
9. $\left\{3^k + \dfrac{1}{3^k}\right\}, \ k \geq 0$
10. $\left\{\alpha^k + \dfrac{1}{\alpha^k}\right\}, \ k \geq 0$
11. $\{\sin 5k\}, \ k \geq 0$
12. $\{\cos 5k\}, \ k \geq 0$
13. $\left\{\sin\left(\dfrac{k\pi}{2} + a\right)\right\}, \ k \geq 0$
14. $\left\{\cos\left(\dfrac{k\pi}{2} + a\right)\right\}, \ k \geq 0$
15. $\sin\left(\dfrac{k\pi}{3} + a\right), \ k \geq 0$
16. $\cos\left(\dfrac{k\pi}{3} + a\right), \ k \geq 0$
17. $\sin\left(\dfrac{k\pi}{6} + a\right), \ k \geq 0$
18. $\cos\left(\dfrac{k\pi}{6} + a\right), \ k \geq 0$
19. $\left\{3^k \cos\left(\dfrac{k\pi}{2} + \dfrac{\pi}{4}\right)\right\}, \ k \geq 0$
20. $\left\{3^k \sin\left(\dfrac{k\pi}{2} + \dfrac{\pi}{4}\right)\right\}, \ k \geq 0$
21. $\{e^{-2k} \cos 3k\}, \ k \geq 0$
22. $\{e^{-3k} \sin 2k\}, \ k \geq 0$

[**Ans. : (1)** $\dfrac{8z - 2z^2}{(3-z)(5-z)}, \ |z| < 3$

(2) $\dfrac{(\alpha+\beta)z - 2z^2}{(\alpha - z)(\beta - z)}, \ |z| <$ min. of $|\alpha|, |\beta|$

(3) $\dfrac{2z^2 - 8z}{(z-3)(z-5)}, \ |z| > 5$

(4) $\dfrac{2z^2 - (\alpha+\beta)z}{(z-\alpha)(z-\beta)}, \ |z| >$ max. of $|\alpha|, |\beta|$

(5) $\dfrac{2z}{(z-2)^2} + \dfrac{3z}{(z-3)^2}, \ |z| > 3$

(6) $\dfrac{\alpha z}{(z-\alpha)^2}+\dfrac{\beta z}{(z-\beta)^2}$, $|z|>$ max. of $|\alpha|, |\beta|$

(7) $-\log\left[\dfrac{(z-2)(z-3)}{z^2}\right]$, $|z|>3$

(8) $-\log\left[\dfrac{(z-\alpha)(z-\beta)}{z^2}\right]$, $|z|>$ max. of $|\alpha|, |\beta|$

(9) $\dfrac{z}{z-3}+\dfrac{3z}{3z-1}$, $|z|>3$

(10) $\dfrac{z}{z-\alpha}+\dfrac{\alpha z}{\alpha z-1}$, $|z|>|\alpha|$

(11) $\dfrac{z\sin 5}{z^2-2z\cos 5+1}$

(12) $\dfrac{z(z-\cos 5)}{z^2-2z\cos 5+1}$

(13) $\dfrac{z^2\sin a+z\cos a}{z^2+1}$, $|z|>1$

(14) $\dfrac{-z\sin a}{z^2+1}$; $|z|>1$

(15) $\dfrac{z}{2}\left[\dfrac{\cos a\cdot\sqrt{3}+\sin a\,(2z-1)}{z^2-z+1}\right]$

(16) $\dfrac{z}{2}\left[\dfrac{\cos a\,(2z-1)-\sin a\cdot\sqrt{3}}{z^2-z+1}\right]$

(17) $\dfrac{z}{2}\left[\dfrac{\cos a\cdot 1+\sin a\,(2z-\sqrt{3})}{z^2-\sqrt{3}\cdot z+1}\right]$

(18) $\dfrac{z}{2}\left[\dfrac{\cos a\,(2z-3\sqrt{3})-\sin a\cdot 1}{z^2-\sqrt{3}\cdot z+1}\right]$

(19) $\dfrac{1}{\sqrt{2}}\cdot\dfrac{z^2-3z}{z^2+9}$, $|z|>3$

(20) $\dfrac{1}{\sqrt{2}}\cdot\dfrac{z^2+3z}{z^2+9}$, $|z|>3$

(21) $\dfrac{z(z-e^{-2}\cos 3)}{z^2-2e^{-2}z\cos 3+e^{-4}}$

(22) $\dfrac{e^{-3}z\sin 2}{z^2-2e^{-3}z\cos 2+e^{-6}}$.]

10. Inverse of Z-transform

We shall now consider the reverse problem *i.e.* given the Z-transform $Z\{f(k)\}=F(z)$ of a sequence to find the original sequence denoted by $\{f(k)\}$ or $Z^{-1}[F(z)]$. We shall consider Z-transforms which are rational functions of z *i.e.* of the form $F(z)=\dfrac{P(z)}{Q(z)}$ and $P(z)$ and $Q(z)$ are algebraic polynomials in z. It should be noted that to find the inverse Z-transform **we should know its region of convergence *i.e.* ROC**. We shall consider the following three methods.

1. Direct Division,
2. Binomial Expansion,
3. Partial Fraction.

(a) Direct Division

In this method, we divide the numerator by the denominator and obtain a power series *i.e.* if $F(z)=\dfrac{P(z)}{Q(z)}$, we actually divide $P(z)$ by $Q(z)$. However, this method, now, is of academic interest only.

Ex. 1 : Find the inverse Z-transform of $\dfrac{1}{z-a}$ if

(i) $|z|>|a|$ **(ii)** $|z|<|a|$.

Sol. : (i) If $|z|>|a|$, $\left|\dfrac{z}{a}\right|>1$ *i.e.* $\left|\dfrac{a}{z}\right|<1$, we consider

$$F(z)=\frac{1}{z-a}$$

By actual division, we obtain a series expansion of $\dfrac{1}{z-a}$

$$\begin{array}{r|l} z-a & 1 \quad \left(\dfrac{1}{z}+\dfrac{a}{z^2}+\dfrac{a^2}{z^3}\right. \\ & 1-\dfrac{a}{z} \\ \hline & \dfrac{a}{z} \\ & \dfrac{a}{z}-\dfrac{a^2}{z^2} \\ \hline & \dfrac{a^2}{z^2} \end{array}$$

$$\therefore \quad \frac{1}{z-a}=\frac{1}{z}+\frac{a}{z^2}+\frac{a^2}{z^3}+\ldots\ldots$$

$$=z^{-1}+az^{-2}+a^2z^{-3}+\ldots\ldots$$

$$=\left\{a^{k-1}\right\}z^{-k}$$

$$\therefore \quad Z^{-1}[F(z)]=\left\{a^{k-1}\right\},\quad k\ge 1$$

(ii) If $|z|<|a|$ *i.e.* $\left|\dfrac{z}{a}\right|<1$, we consider

$$F(z)=\frac{1}{-a+z}$$

$$-a+z \overline{\big) \; 1 \;} \quad \left(-\frac{1}{a}-\frac{z}{a^2}-\frac{z^2}{a^3}\right.$$

$$\begin{array}{l} 1-\frac{z}{a} \\ \hline \frac{z}{a} \\ \frac{z}{a}-\frac{z^2}{a^2} \\ \hline \frac{z^2}{a^2} \\ \frac{z^2}{a^2}-\frac{z^3}{a^3} \\ \hline \frac{z^3}{a^3} \end{array}$$

$$\therefore \quad \frac{1}{-a+z}=-\frac{1}{a}-\frac{z}{a^2}-\frac{z^2}{a^3}$$

$$=-\left\{\frac{1}{a}+\frac{z}{a^2}+\frac{z^2}{a^3}+\ldots\ldots\right\}$$

$$=-\left\{\frac{z^k}{a^{k+1}}\right\}, \quad k \geq 0$$

$$=-\left\{a^{-k-1}\right\} z^k, \quad k \geq 0$$

$$=-\left\{a^{k-1}\right\} z^{-k}, \quad k \leq 0$$

$$\therefore \quad Z^{-1}[F(z)]=-\{a^{k-1}\}, \quad k \leq 0.$$

(b) Binomial Expansion

To apply Binomial Expansion method we take a suitable factor common depending upon ROC from the denominator so that the denominator is of the form $1-r$ where $|r|<1$ and then use Binomial Theorem.

Ex. 1 : Find the inverse Z-transform of $F(z)=\dfrac{1}{z-a}$ when

(i) $|z|<|a|$, **(ii)** $|z|>|a|$.

Sol. : (i) If $|z|<|a|$ *i.e.* $|z/a|<1$ we take *'a'* outside and write

$$F(z)=Z\{f(k)\}=\frac{1}{z-a}=\frac{1}{a[(z/a)-1]}=-\frac{1}{a}\cdot\frac{1}{1-(z/a)}=-\frac{1}{a}\left(1-\frac{z}{a}\right)^{-1}$$

$$=-\frac{1}{a}\left[1+\frac{z}{a}+\frac{z^2}{a^2}+\ldots\ldots+\frac{z^k}{a^k}+\ldots\ldots\right]$$

$$=-\left[\frac{1}{a}+\frac{z}{a^2}+\frac{z^2}{a^3}+\ldots\ldots+\frac{z^k}{a^{k+1}}+\ldots\ldots\right]$$

$$=-[a^{-1}+a^{-2}z+a^{-3}z^2+\ldots\ldots+a^{-k-1}z^k+\ldots\ldots]$$

$\therefore$ Coefficient of $z^k=-a^{-k-1}, \; k \geq 0$

$\therefore$ Coefficient of $z^{-k}=-a^{k-1}, \; k \leq 0$

$\therefore$ $Z^{-1}[F(z)]=f(k)=-a^{k-1}, \; k \leq 0$

(Find $Z\{f(k)\}=-\sum\limits_{k=0}^{\infty} a^{k-1}z^{-k}$ when $f(k)=-a^{k-1}$, $k \leq 0$ and verify the result.)

(ii) If $|z|>|a|$, $\left|\dfrac{z}{a}\right|>1$ *i.e.* $\left|\dfrac{a}{z}\right|<1$, we take *'z'* outside and write

$$F(z)=Z\{f(k)\}=\frac{1}{z-a}=\frac{1}{z[1-(a/z)]}=\frac{1}{z}\left(1-\frac{a}{z}\right)^{-1}$$

$$=\frac{1}{z}\left(1+\frac{a}{z}+\frac{a^2}{z^2}+\ldots\ldots+\frac{a^{k-1}}{z^{k-1}}+\ldots\ldots\right)$$

$$=\frac{1}{z}+\frac{a}{z^2}+\frac{a^2}{z^3}+\ldots\ldots+\frac{a^{k-1}}{z^k}+\ldots\ldots$$

$\therefore$ Coefficient of $z^{-k}=a^{k-1}, \; k \geq 1$

$\therefore$ $Z^{-1}[F(z)]=f(k)=a^{k-1}, \; k \geq 1$

Ex. 2 : Find the inverse Z-transform of $\dfrac{z}{z-a}$, $|z|>a$.

Sol. : We have

$$F(z)=\frac{z}{z-a}=\frac{z}{z[1-(a/z)]}=\left(1-\frac{a}{z}\right)^{-1}$$

$$=1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\ldots\ldots+\left(\frac{a}{z}\right)^k+\ldots.$$

$\therefore$ $Z^{-1}[F(z)]=a^k$; $k \geq 0$, $|z|>a$

Ex. 3 : Find the inverse Z-transform of $\dfrac{z}{z+a}$, $|z|>a$.

Sol. : Changing the sign of a or proceeding as above we get

$$Z^{-1}F(z)=(-a)^k$$

Note : The following inverse Z-transforms may be remembered.

(i) $Z^{-1}\left[\dfrac{1}{z-a}\right] = a^{k-1},\ k \geq 1,\ |z| > a$

(ii) $Z^{-1}\left[\dfrac{1}{z+a}\right] = (-a)^{k-1},\ k \geq 1,\ |z| > a$

(iii) $Z^{-1}\left[\dfrac{z}{z-a}\right] = a^{k},\ k \geq 0,\ |z| > a$

(iv) $Z^{-1}\left[\dfrac{z}{z+a}\right] = (-a)^{k},\ k \geq 0,\ |z| \geq a$

Ex. 4 : Find the inverse Z-transform of $\dfrac{1}{(z-a)^2}$ **(i)** $|z| < a$, **(ii)** $|z| > a$.

Sol. : (i) If $|z| < a$,

$$F(z) = \frac{1}{a^2[1-(z/a)]^2} = \frac{1}{a^2}\left[1-\frac{z}{a}\right]^{-2}$$

$$= \frac{1}{a^2}\left[1+2\cdot\frac{z}{a}+3\cdot\frac{z^2}{a^2}+\ldots..+(n+1)\frac{z^n}{a^n}+\ldots..\right]$$

$$= \frac{1}{a^2}+2\cdot\frac{z}{a^3}+3\cdot\frac{z^2}{a^4}+\ldots..+(n+1)\frac{z^n}{a^{n+2}}+\ldots..$$

Coefficient of $z^n = \dfrac{n+1}{a^{n+2}},\ n \geq 0$

$\therefore$ Coefficient of $z^{-k} = \dfrac{-k+1}{a^{-k+2}},\ -k \geq 0$ *i.e.* $k \leq 0$

$\therefore$ $Z^{-1}[F(z)] = \dfrac{-k+1}{a^{-k+2}},\ k \leq 0$

(ii) If $|z| > a$

$$F(z) = \frac{1}{z^2[1-(a/z)]^2} = \frac{1}{z^2}\left(1-\frac{a}{z}\right)^{-2}$$

$$= \frac{1}{z^2}\left[1+2\cdot\frac{a}{z}+3\cdot\frac{a^2}{z^2}+\ldots..+(n-1)\frac{a^{n-2}}{z^{n-2}}+\ldots..\right]$$

$$= \frac{1}{z^2}+2\cdot\frac{a}{z^3}+3\cdot\frac{a^2}{z^4}+\ldots..+(n-1)\frac{a^{n-2}}{z^n}+\ldots..$$

Coefficient of $z^{-k} = (k-1)\ a^{k-2}$

$\therefore$ $Z^{-1}[F(z)] = (k-1)a^{k-2},\ k \geq 2.$

Ex. 5 : Find the inverse Z-transform of $\dfrac{1}{(z-5)^3}$, $|z| > 5$.

Sol. : Since, $|z| > 5$, $\left|\dfrac{z}{5}\right| > 1$ *i.e.* $\dfrac{5}{|z|} < 1$ hence, we take out 'z' and write.

$$F(z) = \frac{1}{(z-5)^3}$$

$$= \frac{1}{z^3}\cdot\frac{1}{[1-(5/z)]^3} = \frac{1}{z^3}\left(1-\frac{5}{z}\right)^{-3}$$

$$= \frac{1}{z^3}\left[1-(-3)\frac{5}{z}+\frac{(-3)(-4)}{2!}\cdot\left(\frac{5}{z}\right)^2-\frac{(-3)(-4)(-5)}{3!}\cdot\left(\frac{5}{z}\right)^3+\ldots..\right]$$

$$= \frac{1}{z^3}\left[1+3\cdot 5z^{-1}+6\cdot 5^2 z^{-2}+10\cdot 5^3 z^{-3}+\ldots.. + \frac{(n+1)(n+2)}{2}5^n\cdot z^{-n}+\ldots..\right]$$

$$= z^{-3}+3\cdot 5\cdot z^{-4}+6\cdot 5^2 z^{-5}+10\cdot 5^3 z^{-6}+\ldots.. + \frac{(n+1)(n+2)}{2}5^n\cdot z^{-n-3}$$

Coefficient of $z^{-n-3} = \dfrac{(n+1)(n+2)}{2}5^n$ [Put $n+3=k$]

$\therefore$ Coefficient of $z^{-k} = \dfrac{(k-3+1)(k-3+2)}{2}5^{k-3}$

$$= \frac{(k-2)(k-1)}{2}\cdot 5^{k-3}$$

$$Z^{-1}[F(z)] = \frac{(k-2)(k-1)}{2}\cdot 5^{k-3},\ k \geq 3.$$

Ex. 6 : Find the Z-transform of $\dfrac{1}{(z-5)^3}$ if $|z| < 5$.

Sol. : Since $|z| < 5$ we take out 5 and write

$$F(z) = \frac{1}{5^3[(z/5)-1]^3} = -\frac{1}{5^3}\left(1-\frac{z}{5}\right)^{-3}$$

$$= -\frac{1}{5^3}\left[1+3\cdot\frac{z}{5}+6\cdot\frac{z^2}{5^2}+\ldots..+\frac{(n+1)(n+2)}{2}\frac{z^n}{5^n}+\ldots..\right]$$

Coefficient of $z^n = -\dfrac{(n+1)(n+2)}{2}\cdot\dfrac{1}{5^{n+3}},\ n \geq 0$ [Put $n=-k$]

$$\therefore \quad \text{Coefficient of } z^{-k} = -\frac{(-k+1)(-k+2)}{2} \cdot \frac{1}{5^{-k+3}}, \; k \le 0$$

$$\therefore \quad Z^{-1}[F(z)] = -\frac{(-k+1)(-k+2)}{2} \cdot \frac{1}{5^{-k+3}}, \; k \le 0$$

(c) Method of Partial Fractions

If $F(z)$ can be factorised into partial fractions, linear, quadratic or repeated we express $F(z) = \dfrac{P(z)}{Q(z)}$ as the sum such factors, find the constants and then use the method of Binomial Expansion. This is illustrated in the following problems. If the degree of $P(z)$ is greater than that of $Q(z)$ we write $\dfrac{F(z)}{z} = \dfrac{p(z)}{q(z)}$ as in Ex. 3, page 1-48. We now discuss the three cases separately.

(i) Linear non-repeated factors : Let the linear non-repeated factor be $\dfrac{z}{z-a}$. Then

$$Z^{-1}\left(\frac{z}{z-a}\right) = Z^{-1}\left[\frac{1}{1-(a/z)}\right]$$

$$= Z^{-1}\left[1-\left(\frac{a}{z}\right)\right]^{-1}, \quad |z| > |a|$$

$$= Z^{-1}\left[1+\frac{a}{z}+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+.....\right]$$

$$= Z^{-1}\left[1+az^{-1}+a^2z^{-2}+a^3z^{-3}+.....+a^k2^{-k}+.....\right]$$

$$= \{a^k\}$$

$$\text{Also,} \quad Z^{-1}\left(\frac{z}{z-a}\right) = Z^{-1}\left[\frac{z/a}{(z/a)-1}\right]$$

$$= -Z^{-1}\left[\frac{z/a}{1-(z/a)}\right], \quad |a| > |z|$$

$$= -Z^{-1}\left[\left(\frac{z}{a}\right)\left(1-\frac{z}{a}\right)^{-1}\right]$$

$$= -Z^{-1}\left[\frac{z}{a}\left(1+\frac{z}{a}+\frac{z^2}{a^2}+.....\right)\right]$$

$$= -Z^{-1}\left[\left(\frac{z}{a}\right)+\left(\frac{z}{a}\right)^2+\left(\frac{z}{a}\right)^3+.....\right]$$

$$= -Z^{-1}\left\{\frac{z^k}{a^k}\right\}, \quad k \ge 0$$

$$= -Z^{-1}\left\{a^{-k}z^k\right\}, \quad k \ge 0$$

$$= -Z^{-1}\left\{a^k z^{-k}\right\}, \quad k \le 0$$

$$\therefore \quad Z^{-1}\left(\frac{z}{z-a}\right) = -\left\{a^k\right\}, \quad k \le 0$$

Ex. 1 : Find the inverse Z-transform of

$$F(z) = \frac{1}{(z-3)(z-2)}$$

if ROC is (i) $|z| < 2$, (ii) $2 < |z| < 3$, (iii) $|z| > 3$.

Sol. : We have

$$F(z) = \frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2}$$

If $|z| < 2$, clearly $|z| < 3$ $\therefore \left|\dfrac{z}{2}\right| < 1$ and $\left|\dfrac{z}{3}\right| < 1$

Hence, we take out 3 and 2 from the fractions and write.

$$\therefore \quad F(z) = \frac{1}{3[(z/3)-1]} - \frac{1}{2[(z/2)-1]}$$

$$= -\frac{1}{3[1-(z/3)]} + \frac{1}{2[1-(z/2)]}$$

$$= -\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+...+\frac{z^k}{3^k}+...\right)$$

$$+\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+...+\frac{z^k}{2^k}+...\right)$$

$$= -\left(\frac{1}{3}+\frac{z}{3^2}+\frac{z^2}{3^3}+...+\frac{z^k}{3^{k+1}}+...\right)$$

$$+\left(\frac{1}{2}+\frac{z}{2^2}+\frac{z^2}{2^3}+...+\frac{z^k}{2^{k+1}}+...\right)$$

$$= -(3^{-1}+3^{-2}z+3^{-3}z^2+.....+3^{-k-1}z^k+.....)$$

$$+(2^{-1}+2^{-2}z+2^{-3}z^2+.....+2^{-k-1}z^k+.....)$$

From the first series we find that the coefficient of

$$z^k = -3^{-k-1},\ k \geq 0$$

$\therefore$ The coefficient of $z^{-k} = -3^{k-1},\ k \leq 0.$

From the second series, we find that the coefficient of

$$z^k = 2^{-k-1},\ k \geq 0$$

$\therefore$ The coefficient of $z^{-k} = 2^{k-1},\ k \leq 0$

$$\therefore\quad Z^{-1}[F(z)] = -3^{k-1} + 2^{k-1},\ k \leq 0$$

(ii) If $2 < |z| < 3$ *i.e.* $2 < |z| \therefore\ |2/z| < 1$ and $|z| < 3$ *i.e.* $|z/3| < 1$.
Hence, we take out 3 from the first fraction and z from the second fraction.

$$\therefore\quad F(z) = \frac{1}{3[(z/3)-1]} - \frac{1}{z[1-(2/z)]}$$

$$= -\frac{1}{3}\cdot\frac{1}{[1-(z/3)]} - \frac{1}{z}\,\frac{1}{[1-(2/z)]}$$

$$= -\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{3^2}+\ldots+\frac{z^k}{3^k}+\ldots\right)$$

$$-\frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\ldots+\frac{2^{k-1}}{z^{k-1}}+\ldots\right)$$

$$= -\left(\frac{1}{3}+\frac{z}{3^2}+\frac{z^2}{3^3}+\ldots+\frac{z^k}{3^{k+1}}+\ldots\right)$$

$$-\left(\frac{1}{z}+\frac{2}{z^2}+\frac{2^2}{z^3}+\ldots+\frac{2^{k-1}}{z^k}+\ldots\right)$$

$$= -[3^{-1}+3^{-2}z+3^{-3}z^2+\ldots+3^{-k-1}z^k+\ldots]$$

$$-\left[\frac{1}{z}+\frac{2}{z^2}+\ldots+\frac{2^{k-1}}{z^k}+\ldots\right]$$

From the first series we find that the coefficient of

$$z^k = -3^{-k-1},\ k \geq 0$$

$\therefore$ The coefficient of $z^{-k} = -3^{k-1},\ k \leq 0.$

For the second series the coefficient of

$$z^{-k} = -2^{k-1},\ k \geq 1$$

$$\therefore\quad Z^{-1}[F(z)] = -3^{k-1},\ k \leq 0$$

$$= -2^{k-1},\ k \geq 1$$

(iii) If $|z| > 3$, clearly $|z| > 2$ *i.e.* $|z/3| > 1$ and $|z/2| > 1$ *i.e.* $|3/z| < 1$ and $|2/z| < 1$. Hence, we take out z from both fractions.

$$\therefore\quad F(z) = \frac{1}{z[1-(3/z)]} - \frac{1}{z[1-(2/z)]}$$

$$= \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{3}{z}+\frac{3^2}{z^2}+\ldots+\frac{3^{k-1}}{z^{k-1}}+\ldots\right)$$

$$-\frac{1}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\ldots+\frac{2^{k-1}}{z^{k-1}}+\ldots\right)$$

$$= \left(\frac{1}{z}+\frac{3}{z^2}+\frac{3^{k-1}}{z^k}+\ldots\right) - \left(\frac{1}{z}+\frac{2}{z^2}+\ldots+\frac{2^{k-1}}{z^k}+\ldots\right)$$

$\therefore$ Coefficient of $z^{-k} = 3^{k-1} - 2^{k-1},\ k \geq 1.$

$$\therefore\quad Z^{-1}[F(z)] = 3^{k-1} - 2^{k-1},\ k \geq 1$$

$$= 0,\ k \leq 0.$$

Ex. 2 : Find inverse Z-transform of

$$F(z) = \frac{z}{(z-1)(z-2)},\ |z| > 2.$$

Sol. : We have

$$F(z) = \frac{1}{(z-1)(z-2)} = \frac{2}{z-2} - \frac{1}{z-1}$$

Since $|z| > 2$ clearly $|z| > 1$.

$\therefore\ |z/2| > 1$ and $|z| > 1$.

$\therefore\ |2/z| < 1$ and $|1/z| < 1$.

$\therefore$ We take z common.

$$\therefore\quad F(z) = \frac{2}{z[1-(2/z)]} - \frac{1}{z[1-(1/z)]}$$

$$= \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{2}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\ldots+\frac{2^{k-1}}{z^{k-1}}+\ldots\right)$$

$$-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\ldots+\frac{1}{z^{k-1}}+\ldots\right)$$

$$= \left(\frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \dots + \frac{2^k}{z^k} + \dots\right)$$

$$- \left(\frac{1}{z} + \frac{1}{z^2} + \dots + \frac{1}{z^k} + \dots\right)$$

Coefficient of $z^{-k} = 2^k - 1,\ k \geq 1$

$\therefore\ Z^{-1}[F(z)] = 2^k - 1,\ k \geq 1.$

Ex. 3 : Find the inverse z-transform of

$$F(z) = \frac{3z^2 - 18z + 26}{(z-2)(z-3)(z-4)},\quad 3 < z < 4.$$

Sol. : We have (by partial fractions)

$$F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{z-4}$$

$$= \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1} + \frac{1}{z}\left(1 - \frac{3}{z}\right)^{-1} - \frac{1}{4}\left(1 - \frac{z}{4}\right)^{-1}$$

$$= \frac{1}{z}\left(1 + \frac{2}{z} + \dots + \frac{2^{k-1}}{z^{k-1}} + \dots\right)$$

$$+ \frac{1}{z}\left(1 + \frac{3}{z} + \dots + \frac{3^{k-1}}{z^{k-1}} + \dots\right)$$

$$- \frac{1}{4}\left(1 - \frac{z}{4} + \dots + \frac{z^k}{4^k} + \dots\right)$$

From the first series the coefficient of $z^{-k} = 2^{k-1},\ k \geq 0.$

From the second series the coefficient of $z^{-k} = 3^{k-1},\ k \geq 0.$

From the third series the coefficient of $z^k = -\frac{1}{4^{k+1}},\ k \geq 0.$

$\therefore$ The coefficient of $z^{-k} = -4^{k+1},\ k \leq 0$

$$\therefore\ Z^{-1}[F(z)] = \begin{cases} 2^{k-1} + 3^{k-1}, & k \geq 0 \\ -4^{k+1}, & k \leq 0 \end{cases}$$

(ii) Linear repeated factors : When the linear factors are repeated, we use the above technique and expand $\frac{1}{(z-a)^2}, \frac{1}{(z-a)^3}$ by Binomial Theorem as illustrated in the following examples.

Ex. 1 : Find the inverse Z-transform of

$$F(z) = \frac{z+2}{z^2 - 2z + 1},\quad |z| > 1.$$

Sol. : We have

$$F(z) = \frac{z+2}{z^2 - 2z + 1} = \frac{z+2}{(z-1)^2} = \frac{3}{(z-1)^2} + \frac{1}{z-1}$$

Since, $|z| > 1,\ \frac{1}{|z|} < 1.$

$$\therefore\ F(z) = \frac{3}{z^2[1 - (1/z)]^2} + \frac{1}{z[1 - (1/z)]}$$

$$= \frac{3}{z^2}\left(1 - \frac{1}{z}\right)^{-2} + \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{3}{z^2}\left(1 - (-2)\cdot\frac{1}{z} + \frac{(-2)(-3)}{2!}\cdot\frac{1}{z^2} - \frac{(-2)(-3)(-4)}{3!}\cdot\frac{1}{z^3} + \dots\right)$$

$$+ \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots\right)$$

$$= \frac{3}{z^2}\left(1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \dots + \frac{k-1}{z^{k-2}} + \dots\right)$$

$$+ \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \dots + \frac{1}{z^{k-1}} + \dots\right)$$

$$= 3\left(\frac{1}{z^2} + \frac{2}{z^3} + \dots + \frac{k-1}{z^k} + \dots\right) + \left(\frac{1}{z} + \frac{1}{z^2} + \dots + \frac{1}{z^k}\right)$$

$$= \frac{1}{z} + \frac{4}{z^2} + \frac{7}{z^3} + \dots + \frac{3k-2}{z^k} + \dots$$

$\therefore$ Coefficient of $z^{-k} = 3k - 2,\ k \geq 1.$

$\therefore\ Z^{-1}[F(z)] = 3k - 2,\ k \geq 1.$

Ex. 2 : Find the inverse Z-transform of

$$\frac{2z^2 - 10z + 13}{(z-3)^2(z-2)},\quad 2 < |z| < 3.$$

Sol. : We have

$$F(z) = \frac{2z^2 - 10z + 13}{(z-3)^2(z-2)} = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

Since, $2 < |z|,\ |2/z| < 1$ and since $|z| < 3,\ |z/3| < 1.$

$$\therefore\ F(z) = \frac{1}{z}\cdot\frac{1}{[1 - (2/z)]} + \frac{1}{3}\cdot\frac{1}{[(z/3) - 1]} + \frac{1}{9}\cdot\frac{1}{[(z/3) - 1]^2}$$

$$= \frac{1}{z}\cdot\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1 - \frac{z}{3}\right)^{-2}$$

$$= \frac{1}{z} \cdot \left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \ldots\ldots + \frac{2^{k-1}}{z^{k-1}} + \ldots\ldots\right)$$

$$- \frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \ldots\ldots + \frac{z^k}{3^k} + \ldots\ldots\right)$$

$$+ \frac{1}{9}\left(1 + 2 \cdot \frac{z}{3} + 3 \cdot \frac{z^2}{3^2} + \ldots\ldots + (k+1)\frac{z^k}{3^k} + \ldots\ldots\right)$$

$\therefore$ From the first series we find that the coefficient of z^{-k} is 2^{k-1}, from the second series the coefficient of z^k is $-\dfrac{1}{3^{k+1}}$ and from the third series the coefficient of z^k is $\dfrac{k+1}{3^{k+2}}$.

$\therefore$ From the first series, coefficient of $z^{-k} = 2^{k-1}$, $k \geq 1$.

From second and third series,

$$\text{Coefficient of } z^k = \frac{k+1}{3^{k+2}} - \frac{1}{3^{k+1}} = \frac{k-2}{3^{k+2}}, \; k \geq 0$$

$$\therefore \text{Coefficient of } z^{-k} = \frac{-k-2}{3^{-k+2}}, \; k \leq 0.$$

Hence, $Z^{-1}[F(z)] = 2^{k-1}$, $k \geq 1$

$$= \frac{-k-2}{3^{-k+2}}, \; k \leq 0.$$

Ex. 3 : Find the inverse Z-transform of

$$F(z) = \frac{z^2}{[z - (1/4)][z - (1/5)]}$$

(i) $\dfrac{1}{5} < |z| < \dfrac{1}{4}$, **(ii)** $|z| < \dfrac{1}{5}$.

Sol. : (i) Since, the degree of the numerator is equal to the degree of the numerator, we write,

$$\frac{F(z)}{z} = \frac{z}{[z - (1/4)][z - (1/5)]}$$

$$= \frac{5}{z - (1/4)} - \frac{4}{z - (1/5)}$$

Now, $\dfrac{1}{5} < |z|$ $\therefore$ $\dfrac{1}{|5z|} < 1$; $|z| < \dfrac{1}{4}$ $\therefore$ $|4z| < 1$.

$$\therefore \frac{F(z)}{z} = \frac{5 \cdot 4}{4z - 1} - \frac{4}{z[1 - (1/5z)]}$$

$$= -\frac{20}{1 - 4z} - \frac{4}{z[1 - (1/5z)]}$$

$$= -20(1 - 4z)^{-1} - \frac{4}{z}\left(1 - \frac{1}{5z}\right)^{-1}$$

$$= -20(1 + 4z + (4z)^2 + \ldots\ldots) - \frac{4}{z}\left(1 + \frac{1}{5z} + \frac{1}{(5z)^2} + \ldots\ldots\right)$$

$$\therefore \frac{F(z)}{z} = -5(4 + 4^2 z + \ldots\ldots + 4^k z^{k-1} + \ldots\ldots)$$

$$- \frac{4}{z}\left(1 + \frac{1}{5z} + \ldots\ldots + \frac{1}{5^k} \cdot \frac{1}{z^k} + \ldots\ldots\right)$$

$$\therefore F(z) = -5(4z + 4^2 z^2 + \ldots\ldots + 4^k z^k + \ldots\ldots)$$

$$-4\left(1 + \frac{1}{5z} + \ldots\ldots + \frac{1}{5^k} \cdot \frac{1}{z^k} + \ldots\ldots\right)$$

$$= -5 \cdot \sum_{k=1}^{\infty} 4^k \cdot z^k - 4 \cdot \sum_{k=0}^{\infty} \frac{1}{5^k} \cdot \frac{1}{z^k}$$

Since, we want the coefficient of z^{-k} we change k to $-k$ in the first term.

$$\therefore F(z) = -5 \cdot \sum_{k=-1}^{-\infty} (1/4)^k \cdot z^{-k} - 4 \cdot \sum_{k=0}^{\infty} (1/5^k) \cdot z^{-k}$$

$$\therefore Z^{-1}[F(z)] = \underset{k < 0}{-5 \cdot (1/4)^k} \; \underset{k \geq 0}{- 4 \cdot (1/5^k)}$$

(ii) Since, $|z| < 1/5$, clearly $|z| < 1/4$ $\therefore$ $|5z| < 1$ and $|4z| < 1$.

$$\therefore \frac{F(z)}{z} = \frac{5}{z - (1/4)} - \frac{4}{z - (1/5)}$$

$$= \frac{20}{4z - 1} - \frac{20}{5z - 1}$$

$$= -\frac{20}{1 - 4z} + \frac{20}{1 - 5z}$$

$$= 20\left[\frac{1}{1 - 5z} - \frac{1}{1 - 4z}\right]$$

$$= 20[(1 - 5z)^{-1} - (1 - 4z)^{-1}]$$

$$= 20[(1 + 5z + \ldots + 5^{k-1} z^{k-1} + \ldots)$$

$$- (1 + 4z + \ldots + 4^{k-1} z^{k-1} + \ldots)]$$

$$\therefore \frac{F(z)}{z} = 4 \cdot (5 + 5^2 \cdot z + \ldots + 5^k z^{k-1} + \ldots)$$

$$- 5(4 + 4^2 \cdot z + \ldots + 4^k \cdot z^{k-1} + \ldots)$$

$$\therefore \quad F(z) = 4(5z + 5^2 z^2 + \ldots + 5^k z^k + \ldots)$$
$$- 5(4z + 4^2 z^2 + \ldots + 4^k z^k + \ldots)$$
$$= 4 \sum_{k=1}^{\infty} 5^k \cdot z^k - 5 \cdot \sum_{k=1}^{\infty} 4^k \cdot z^k$$

Since, we want the coefficient of z^{-k} we replace k by $-k$.

$$\therefore \quad F(z) = 4 \sum_{k=-1}^{-\infty} 5^{-k} \cdot z^{-k} - 5 \sum_{k=-1}^{-\infty} 4^{-k} \cdot z^{-k}$$

$$\therefore \quad Z^{-1}[F(z)] = \underset{k<0}{4 \cdot 5^{-k}} - \underset{k<0}{5 \cdot 4^{-k}}$$

EXERCISE

Find the inverse Z-transforms of the following :

1. $\dfrac{1}{z-1}$, $|z| < 1$, $|z| > 1$
2. $\dfrac{1}{z-3}$, $|z| < 3$, $|z| > 3$
3. $\dfrac{z}{z-1}$, $|z| < 1$, $|z| > 1$
4. $\dfrac{z}{z-a}$, $|z| < a$, $|z| > a$, $a > 0$
5. $\dfrac{1}{(z-1)^2}$, $|z| < 1$, $|z| > 1$
6. $\dfrac{1}{(z-5)^2}$, $|z| < 5$, $|z| > 5$
7. $\dfrac{1}{(z-3)^2}$, $|z| < 3$, $|z| > 3$
8. $\dfrac{1}{(z-1)^3}$, $|z| < 1$, $|z| > 1$
9. $\dfrac{3z^2 + 2z}{z^2 - 3z + 2}$, $1 < |z| < 2$
10. $\dfrac{1}{z^2 - 3z + 2}$, $|z| > 2$
11. $\dfrac{z}{[z-(1/4)][z-(1/5)]}$, $\dfrac{1}{5} < |z| < \dfrac{1}{4}$
12. $\dfrac{z}{(z-2)(z-3)}$, $|z| < 2$, $2 < |z| < 3$, $|z| > 3$
13. $\dfrac{1}{[z-(1/2)][z-(1/3)]}$ (i) $\dfrac{1}{3} < |z| < \dfrac{1}{2}$, (ii) $\dfrac{1}{2} < |z|$
14. $\dfrac{z^3}{(z-1)(z-2)^2}$, $|z| > 2$
15. $\dfrac{z^3}{(z-3)(z-2)^2}$, $|z| > 3$

[**Ans. :** **(1)** (a) -1, $k \le 0$; (b) 1, $k \ge 1$,
(2) (a) -3^{k-1}, $k \le 0$; (b) 3^{k-1}, $k \ge 1$,
(3) (a) -1, $k < 0$; (b) 1, $k \ge 0$,
(4) (a) $-a^k$, $k < 0$; (b) a^k, $k \ge 0$,
(5) (a) $-k+1$, $k \le 0$; (b) $k-1$, $k \ge 2$,

(6) (a) $\dfrac{-k+1}{5^{-k+2}}$, $k \le 0$; (b) $(k-1)\, 5^{k-2}$, $k \ge 2$,
(7) (a) $-\dfrac{(-k+1)(-k+2)}{2} \cdot \dfrac{1}{3^{-k+3}}$, $k \le 0$;
(b) $\dfrac{(k-2)(k-1)}{2} \cdot 3^{k-3}$, $k \ge 3$,
(8) (a) $-\dfrac{(-k+1)(-k+2)}{2}$, $k \le 0$;
(b) $\dfrac{(k-2)(k-1)}{2}$, $k \ge 3$,
(9) $-5 - 8\,(2)^k$,
$(k \ge 0)$ $(k < 0)$,
(10) $2^{k-1} - 1$, $k \ge 1$,
(11) $5\left(\dfrac{1}{4}\right)^k + 4\left(\dfrac{1}{5}\right)^k$,
$(k \le 0)$, $(k \ge 0)$,
(12) (i) $2^k - 3^k$, $k \le 0$, (ii) -2^k, $(k > 0)$, $-3^k (k \le 0)$,
(iii) $3^k - 2^k$, $k \ge 0$,
(13) (a) $f(k) = -\dfrac{6}{3^{k-1}}$, $k > 0$;
$= -12 \cdot 2^{-k}$, $k \le 0$
(b) $f(k) = 6\left[\dfrac{1}{2^{k-1}} - \dfrac{1}{3^{k-1}}\right]$, $k \ge 1$
(14) Hint : Write $\dfrac{F(z)}{z} = \dfrac{1}{(z-1)} + \dfrac{4}{(z-2)^2}$

$$= \frac{1}{z} \cdot \left(1 - \frac{1}{z}\right)^{-1} + \frac{4}{z^2}\left(1 - \frac{2}{z}\right)^{-2}$$

$$F(z) = 1 + \frac{1+2^2}{z} + \frac{1+2 \cdot 2^3}{z^2} + \frac{1+3 \cdot 2^4}{z^3} + \ldots$$

$$\therefore \quad f(k) = 1 + k \cdot 2^{k+1},\ k \ge 0$$

(15) Hint : Write $\dfrac{F(z)}{z} = \dfrac{9}{(z-3)} - \dfrac{8}{(z-2)} - \dfrac{4}{(z-2)^2}$

$$= \frac{9}{z}\left(1 - \frac{3}{z}\right)^{-1} - \frac{8}{z}\left(1 - \frac{2}{z}\right)^{-1} - \frac{4}{z^2}\left(1 - \frac{2}{z}\right)^{-2}$$

$$f(k) = 3^{k+2} - 2^{k+3} - k \cdot 2^{k+1},\ k \ge 0\ ]$$

(iii) Non-repeated Quadratic Factor : Let the non-repeated quadratic factor be

$$\frac{Mz^2+Nz}{z^2+pz+q} \quad \text{.......... (1)}$$

We know that,

$$Z\{c^k \cos\alpha k\} = \frac{z^2 - cz\cos\alpha}{z^2 - 2cz\cos\alpha + c^2} \quad \text{.......... (2)}$$

and $$Z\{c^k \sin\alpha k\} = \frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2} \quad \text{.......... (3)}$$

To find $$Z^{-1}\left\{\frac{Mz^2+Nz}{z^2+pz+q}\right\} \quad \text{.......... (4)}$$

We shall use

$$Z^{-1}\left\{\frac{z^2 - cz\cos\alpha}{z^2 - 2cz\cos\alpha + c^2}\right\} \quad \text{.......... (5)}$$

and $$Z^{-1}\left\{\frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2}\right\} \quad \text{.......... (6)}$$

In view of this, we write

$$Mz^2 = M[z^2 - cz\cos\alpha] + Mcz\cos\alpha$$

and $$Nz = \frac{N\,cz\sin\alpha}{c\sin\alpha}\ ;\ p = -2c\cos\alpha,\ q = c^2.$$

Then (5) and (6) can be used to find (4).

This means we write

$$\frac{Mz^2+N}{z^2+pz+q} = \frac{M(z^2 - cz\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2} + \frac{\dfrac{(M\,cz\cos\alpha + N)}{c\sin\alpha}\cdot cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2} \quad \text{......... (A)}$$

Since we are putting $p = -2c\cos\alpha$ and $q = c^2$, we have

$$\frac{p}{-2c} = \cos\alpha \quad \therefore \left|\frac{p}{2c}\right| < 1.$$

Hence, we can use (A) if $\left|\frac{p}{2c}\right| < 1$.

If $\left|\frac{p}{2c}\right| > 1$, reasoning as above, we write

$$\frac{Mz^2+Nz}{z^2+pz+q} = \frac{M(z^2 - cz\cosh\alpha)}{z^2 - 2cz\cosh\alpha + c^2} + \frac{\dfrac{(Mc\cosh\alpha + N)}{c\sinh\alpha}\cdot cz\sinh\alpha}{z^2 - 2cz\cosh\alpha + c^2}$$

Ex. 1 : Find $Z^{-1}\left(\frac{z^2}{z^2+1}\right)$, $|z| > 1$.

Sol. : Since $p = 0$, $c = 1$, $\left|\frac{p}{2c}\right| < 1$.

Comparing $z^2 + 1$ with $z^2 - 2\,cz\cos\alpha + c^2$, we see that $c = 1$,

$$\cos\alpha = 0 \quad \therefore \alpha = \frac{\pi}{2}$$

Now, we write

$$\frac{z^2}{z^2+1} = \frac{(z^2 - cz\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2} + \frac{\left(\dfrac{2c\cos\alpha}{c\sin\alpha}\right)\cdot cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2} \quad \text{.......... (A)}$$

Putting $c = 1$, $\alpha = \frac{\pi}{2}$ *i.e.* $\cos\frac{\pi}{2} = 0$ in the bracketed quantity of the second term, we find that $\frac{2c\cos\alpha}{c\sin\alpha} = 0$.

$$\therefore \quad \frac{z^2}{z^2+1} = \frac{z^2 - cz\cos\alpha}{z^2 - 2cz\cos\alpha + c^2}$$

$$\therefore \quad Z^{-1}\left(\frac{z^2}{z^2+1}\right) = Z^{-1}\left(\frac{z^2 - cz\cos\alpha}{z^2 - 2cz\cos\alpha + c^2}\right)$$

$$= \left\{c^k \cos\alpha k\right\}$$

Putting $c = 1$, $\alpha = \frac{\pi}{2}$, we get

$$Z^{-1}\left(\frac{z^2}{z^2+1}\right) = \left\{\cos\frac{\pi}{2}k\right\}.$$

Ex. 2 : Find $Z^{-1}\left(\frac{2z^2+3z}{z^2+z+1}\right)$, $|z| > 1$.

Sol. : Since $p = 1$, $c = 1$, $\left|\frac{p}{2c}\right| < \left|\frac{1}{z}\right| < 1$.

Comparing $z^2 + z + 1$ with $z^2 - 2\,cz\cos\alpha + c^2$, we see that $c = 1$ and $-2\cos\alpha = 1 \quad \therefore \cos\alpha = -1/2$.

As $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, $\alpha = \frac{2\pi}{3}$.

Now we write

$$\frac{2z^2+3z}{z^2+z+1} = \frac{2(z^2 - cz\cos\alpha)}{z^2 - 2cz\cos\alpha + c^2} + \frac{\left(\dfrac{2c\cos\alpha+3}{c\sin\alpha}\right)\cdot cz\sin\alpha}{z^2-2cz\cos\alpha+c^2} \qquad \text{......... (A)}$$

Putting $c = 1$, $\alpha = \frac{2\pi}{3}$ *i.e.* $\cos\alpha = -\frac{1}{2}$, $\sin\alpha = \frac{\sqrt{3}}{2}$ in the bracketed quantity of the second term, we find that

$$\frac{2c\cos\alpha+3}{c\sin\alpha} = \frac{2(1)(-1/2)+3}{1\cdot(\sqrt{3}/2)} = \frac{2}{\sqrt{3}/2} = \frac{4}{\sqrt{3}}.$$

Hence, from (A) we get

$$\frac{2z^2+3z}{z^2+z+1} = \frac{2(z^2-cz\cos\alpha)}{z^2-2cz\cos\alpha+c^2} + \frac{4}{\sqrt{3}}\cdot\frac{cz\sin\alpha}{z^2-2cz\cos\alpha+c^2}$$

Taking inverse Z-transforms of both sides,

$$Z^{-1}\left(\frac{2z^2+3z}{z^2+z+1}\right) = 2\cdot Z^{-1}\left(\frac{z^2-cz\cos\alpha}{z^2-2cz\cos\alpha+c^2}\right) + \frac{4}{\sqrt{3}}\cdot Z^{-1}\left(\frac{cz\sin\alpha}{z^2-2cz\cos\alpha+c^2}\right)$$

$$= 2\left\{c^k\cos\alpha k\right\} + \frac{4}{\sqrt{3}}\left\{c^k\sin\alpha k\right\}$$

Putting $c = 1$, $\alpha = \frac{2\pi}{3}$, we get

$$Z^{-1}\left(\frac{2z^2+3z}{z^2+z+1}\right) = 2\left\{\cos\left(\frac{2\pi}{3}\right)k\right\} + \frac{4}{\sqrt{3}}\left\{\sin\left(\frac{2\pi}{3}\right)k\right\}, \quad |z| > 1.$$

Ex. 3 : Find $Z^{-1}\left(\frac{3z^2+4z}{z^2-z+1}\right)$, $|z| > 1$.

Sol. : Since $p = -1$, $c = 1$, $\left|\frac{p}{2c}\right| < 1$.

Comparing $z^2 - z + 1$ with $z^2 - 2\,cz\cos\alpha + c^2$, we see that $c = 1$, $2\cos\alpha = 1$ *i.e.* $\cos\alpha = 1/2$ $\therefore$ $\alpha = \pi/3$.

Now, we write

$$\frac{3z^2+4z}{z^2-z+1} = \frac{3(z^2-cz\cos\alpha)}{z^2-2cz\cos\alpha+c^2} + \frac{\left(\dfrac{3c\cos\alpha+4}{c\sin\alpha}\right)cz\sin\alpha}{z^2-2cz\cos\alpha+c^2} \qquad \text{......... (A)}$$

Putting $c = 1$, $\alpha = \frac{\pi}{3}$, $\cos\alpha = \frac{1}{2}$ and $\sin\alpha = \frac{\sqrt{3}}{2}$ in the bracketed quantity of the second term, we find that

$$\frac{3c\cos\alpha+4}{c\sin\alpha} = \frac{(3)(1/2)+4}{(1)(\sqrt{3}/2)} = \frac{11}{\sqrt{3}}$$

Hence, from (A), we get,

$$\frac{3z^2+4z}{z^2-z+1} = 3\cdot\frac{(z^2-cz\cos\alpha)}{z^2-2cz\cos\alpha+c^2} + \frac{11}{\sqrt{3}}\cdot\frac{cz\sin\alpha}{z^2-2cz\cos\alpha+c^2}$$

Taking inverse Z-transforms of both sides,

$$Z^{-1}\left(\frac{3z^2+4z}{z^2-z+1}\right) = 3\cdot\left\{c^k\cos\alpha k\right\} + \frac{11}{\sqrt{3}}\left\{c^k\sin\alpha k\right\}$$

Putting $c = 1$, $\alpha = \frac{\pi}{3}$, we get

$$Z^{-1}\left(\frac{3z^2+4z}{z^2-z+1}\right) = 3\left\{\cos\left(\frac{\pi k}{3}\right)\right\} + \frac{11}{\sqrt{3}}\left\{\sin\left(\frac{\pi k}{3}\right)\right\}$$

Ex. 4 : Find $Z^{-1}\left(\frac{3z^2+4}{z^2+z+(1/9)}\right)$, $|z| > \frac{1}{3}$.

Sol. : Since $c^2 = \frac{1}{9}$, $p = 1$, $\left|\frac{p}{2c}\right| = \frac{3}{2} > 1$.

Hence, we compare $z^2 + z + \left(\frac{1}{9}\right)$ with $z^2 - 2\,cz\cosh\alpha + c^2$.

$$\therefore \quad c = -\frac{1}{3}, \quad -2c\cosh\alpha = 1$$

$$\therefore \quad \cosh\alpha = \frac{1}{-2c} = \frac{3}{2},$$

$$\sinh\alpha = \sqrt{\cos^2\alpha - 1}$$

$$\therefore \quad \sinh\alpha = \frac{\sqrt{5}}{2}.$$

Hence, we write

$$\frac{3z^2+4}{z^2+z+(1/9)} = \frac{3(z^2-cz\cos h\,\alpha)}{z^2-2cz\cos h\,\alpha+c^2} + \frac{\left(\dfrac{3c\cos h\,\alpha+4}{c\sin h\,\alpha}\right)\cdot cz\sin h\,\alpha}{z^2-2cz\cos h\,\alpha+c^2} \quad \text{.......... (A)}$$

Putting $\cos h\,\alpha = \frac{3}{2}$, $\sin h\,\alpha = \frac{\sqrt{5}}{2}$, $c = -\frac{1}{3}$ in the bracketed quantity of the second term of (A), we get,

$$\frac{3c\cos h\,\alpha+4}{c\sin h\,\alpha} = \frac{3\cdot(-1/3)\cdot(3/2)+4}{(-1/3)\cdot(\sqrt{5}/2)} = -\frac{15}{\sqrt{5}}.$$

$$\therefore \quad \frac{3z^2+4}{z^2+z+(1/9)} = \frac{3(z^2-cz\cos h\,\alpha)}{z^2-2cz\cos h\,\alpha+c^2} - \frac{15}{\sqrt{5}}\cdot\frac{cz\sin h\,\alpha}{z^2-2cz\cos h\,\alpha+c^2}$$

Taking inverse Z-transform of both sides,

$$Z^{-1}\left(\frac{3z^2+4}{z^2+z+(1/9)}\right) = 3Z^{-1}\left(\frac{z^2-cz\cos h\,\alpha}{z^2-2cz\cos h\,\alpha+c^2}\right) - \sqrt{\frac{5}{3}}\cdot Z^{-1}\left(\frac{cz\sin h\,\alpha}{z^2-2cz\cos h\,\alpha+c^2}\right)$$

$$= 3\left\{c^k\cos h\,\alpha\right\} - \sqrt{\frac{5}{3}}\cdot\left\{c^k\sin h\,\alpha\,k\right\}$$

$$= 3\left\{\left(-\frac{1}{3}\right)^k\cos h\,\alpha\,k\right\} - \sqrt{\frac{5}{3}}\left\{\left(\frac{1}{3}\right)^k\sin h\,\alpha\,k\right\}, \quad k \ge 0.$$

EXERCISE

Find the inverse Z-transforms of the following.

1. $\dfrac{z^2+z}{z^2+z+1}$, $|z| > 1$
2. $\dfrac{2z^2+3z}{z^2+z+(1/16)}$, $|z| > 2+\sqrt{3}$
3. $\dfrac{2z^2+3z}{z^2+z+(1/9)}$, $|z| > \dfrac{1}{3}$
4. $\dfrac{2z^2+3z}{z^2-z+1}$, $|z| > 1.$

[**Ans. :** **(1)** $\left\{\frac{1}{\sqrt{3}}\sin\frac{2\pi k}{3} + \cos\frac{2\pi k}{3}\right\}$, $k \ge 0$

(2) $\left\{2\left(-\frac{1}{4}\right)^k\cos h\,\alpha\,k - \frac{16}{\sqrt{3}}\left(-\frac{1}{4}\right)^k\sin h\,\alpha\,k\right\}$, $k \ge 0$

(3) $\left\{2\cdot\left(-\frac{1}{3}\right)^k\cos h\,\alpha\,k - \frac{12}{\sqrt{5}}\left(-\frac{1}{3}\right)^k\right\}\sin h\,\alpha\,k$, $k \ge 0$

(4) $\left\{2\cos\frac{\pi k}{3} + \frac{8}{\sqrt{3}}\sin\frac{\pi k}{3}\right\}$, $k \ge 0.$]

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$$\therefore \int_0^\infty \frac{\sin\omega \cdot \cos\omega x}{\omega} d\omega = \frac{\pi}{2}\cdot f(x)$$

$$= \begin{cases} \frac{\pi}{2} \text{ for } f(x)=1 \text{ when } |x|<1 \\ 0 \text{ for } f(x)=0 \text{ when } |x|>1 \end{cases}$$

At $|x| = 1$ *i.e.* $x = \pm 1$, $f(x)$ is discontinuous and the integral

$$= \frac{\pi}{2}\cdot\frac{1}{2}\left\{\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right\}$$

$$= \frac{\pi}{4}[1+0] = \frac{\pi}{4}.$$

Ex. 2 : Expressing the above function as Fourier Integral, evaluate

$$\int_0^\infty \frac{\sin\omega}{\omega} d\omega$$

Sol. : In the final result obtained above put $x = 0$.

$$\therefore \int_0^\infty \frac{\sin\omega}{\omega} d\omega = \frac{\pi}{2} f(0) = \frac{\pi}{2} \qquad [\because f(0) = 1]$$

Note : Unfortunately there is no uniformity in the notation of Fourier Integral and Fourier transforms. Some authors use λ or α in place of ω and t in place of s.

Ex. 3 : Find the Fourier Integral representation of

$$f(x) = \begin{cases} 0, & x<0 \\ 1/2, & x=0 \\ e^{-x}, & x>0 \end{cases}$$

Sol. : The Fourier Integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty \int_{-\infty}^\infty f(s)\cos\omega(s-x)\, d\omega\, ds\right]$$

$$= \frac{1}{\pi}\left[\int_0^\infty \int_{-\infty}^0 0\, d\omega\, ds + \int_0^\infty \int_0^\infty e^{-s}\cos\omega(s-x)\, d\omega\, ds\right]$$

$$= \frac{1}{\pi}\int_0^\infty \int_0^\infty e^{-s}(\cos\omega s\cos\omega x + \sin\omega s\sin\omega x)\, d\omega\, ds$$

$$= \frac{1}{\pi}\int_0^\infty \left\{\cos\omega x\int_0^\infty e^{-s}\cos\omega s\, ds + \sin\omega x\int_0^\infty e^{-s}\sin\omega s\, ds\right\} d\omega$$

But $\int e^{ax}\cos bx\, dx = \frac{1}{a^2+b^2}\left[e^{ax}(a\cos bx + b\sin bx)\right]$

and $\int e^{ax}\sin bx\, dx = \frac{1}{a^2+b^2}\left[e^{ax}(a\sin bx - b\cos bx)\right]$

$$\therefore f(x) = \frac{1}{\pi}\int_0^\infty \left\{\cos\omega x\left[\frac{1}{1+\omega^2}e^{-s}(-\cos\omega s + \omega\sin\omega s)\right]_0^\infty + \sin\omega x\left[\frac{1}{1+\omega^2}e^{-s}(-\sin\omega s - \omega\cos\omega s)\right]_0^\infty\right\} d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \left(\frac{\cos\omega x}{1+\omega^2} + \frac{\omega\sin\omega x}{1+\omega^2}\right) d\omega$$

$$\therefore f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega$$

And when $x = 0$,

$$f(0) = \frac{1}{\pi}\int_0^\infty \frac{1}{1+\omega^2} d\omega = \frac{1}{\pi}\left[\tan^{-1}\omega\right]_0^\infty$$

$$= \frac{1}{\pi}\cdot\frac{\pi}{2} = \frac{1}{2}.$$

Hence, the Fourier Integral representation of $f(x)$ is

$$f(x) = \begin{cases} 0, & x<0 \\ 1/2, & x=0 \\ \frac{1}{\pi}\int_0^\infty \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2} d\omega, & x>0 \end{cases}$$

Ex. 4 : Express the function

$$f(x) = \begin{cases} -e^{kx} & \text{for } x<0 \\ e^{-kx} & \text{for } x>0 \end{cases}$$

as Fourier Integral and hence, prove that

$$\int_0^\infty \frac{\omega\sin\omega x}{\omega^2+k^2} d\omega = \frac{\pi}{2}e^{-kx} \text{ if } x>0,\ k>0.$$

Sol. : The Fourier Integral for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(s)\cos\omega(s-x)\, d\omega\, ds$$

But since the given function $f(x)$ is an odd function we use (3), of § 3(b).

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\infty e^{-ks}\sin\omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left[\frac{1}{k^2+\omega^2}e^{-ks}(-k\sin\omega s - \omega\cos\omega s)\right]_0^\infty d\omega$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \cdot \frac{\omega}{k^2+\omega^2} d\omega$$

$$\therefore \int_0^{\infty} \frac{\omega \sin \omega x}{\omega^2 + k^2} d\omega = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-kx} \text{ if } x > 0.$$

Ex. 5 : Find the Fourier integral representation of

$$f(x) = e^{-|x|}, \quad -\infty < x < \infty$$

Sol. : The Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\left[\int_0^{\infty} \int_{-\infty}^{\infty} f(s) \cos \omega (s - x) d\omega \, ds\right]$$

Since $f(x)$ is an even function, we have by (2) of § 3 (b)

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \omega x \int_0^{\infty} f(s) \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \omega x \int_0^{\infty} e^{-s} \cos \omega s \, ds \, d\omega$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \omega x \left[\frac{e^{-s}}{1+\omega^2}(-\cos \omega s + \omega \sin \omega s)\right]_0^{\infty} d\omega$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \omega x \left[0 + \frac{1}{1+\omega^2}\right] d\omega$$

$$\therefore \quad f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\cos \omega x}{1+\omega^2} d\omega .$$

Ex. 6 : Express the function

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & 0 \end{cases}$$

as Fourier sine integral and evaluate $\int_0^{\infty} \frac{\sin \omega x \cdot \sin \pi\omega}{1-\omega^2} d\omega$.

Sol. : The Fourier Sine Integral of $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \omega x \int_0^{\infty} f(s) \sin \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \omega x \int_0^{\pi} \sin s \sin \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \omega x \cdot \left(-\frac{1}{2}\right) \int_0^{\pi} [\cos s (1+\omega) - \cos s (1-\omega)] \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \omega x \left(-\frac{1}{2}\right)\left[\frac{\sin s (1+\omega)}{1+\omega} - \frac{\sin s (1-\omega)}{1-\omega}\right]_0^{\pi} d\omega$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \omega x \cdot \left(-\frac{1}{2}\right)\left[-\frac{2 \sin \pi\omega}{1-\omega^2}\right] d\omega$$

$$[\because \sin(\pi + \theta) = -\sin \theta \text{ and } \sin(\pi - \theta) = \sin \theta]$$

$$\therefore \quad f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin \omega x \sin \pi\omega}{1-\omega^2} dx$$

$$\therefore \quad \int_0^{\infty} \frac{\sin \omega x \cdot \sin \pi\omega}{1-\omega^2} dx = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & 0 \end{cases}$$

Ex. 7 : Express the function

$$f(x) = \begin{cases} \pi/2 & \text{for } 0 < x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

as Fourier sine Integral and show that

$$\int_0^{\infty} \frac{1-\cos \pi\omega}{\omega} \sin \omega x \, d\omega = \frac{\pi}{2} \text{ when } 0 < x < \pi.$$

Sol. : Fourier sine integral is,

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \omega x \int_0^{\pi} \frac{\pi}{2} \cdot \sin \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \omega x \cdot \frac{\pi}{2}\left[-\frac{\cos \omega s}{\omega}\right]_0^{\pi} d\omega$$

$$= \int_0^{\infty} \sin \omega x \left[\frac{-\cos \pi\omega + 1}{\omega}\right] d\omega$$

$$= \int_0^{\infty} \frac{1-\cos \pi\omega}{\omega} \cdot \sin \omega x \, d\omega$$

$$\therefore \quad \int_0^{\infty} \frac{1-\cos \pi\omega}{\omega} \cdot \sin \omega x \cdot d\omega = f(x) = \frac{\pi}{2} \text{ when } 0 < x < \pi.$$

Ex. 8 : Find the Fourier Cosine integral representation of the function

$$f(x) = \begin{cases} x^2, & 0 < x < a \\ 0, & x > a \end{cases}$$

Sol. : Fourier cosine integral representation of $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \omega x \int_0^{\infty} f(s) \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \omega x \int_0^{a} s^2 \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \omega x \left[s^2\left(\frac{\sin \omega s}{\omega}\right) - \left(-\frac{\cos \omega s}{\omega^2}\right)(2s) + \left(-\frac{\sin \omega s}{\omega^n}\right)(2)\right]_0^{a} d\omega$$

$$= \frac{2}{\pi} \int_0^{\infty} \cos \omega \left[\frac{a^2 \sin a\omega}{\omega} + \frac{2a \cos a\omega}{\omega^2} - \frac{2 \sin a\omega}{\omega^3}\right] d\omega$$

Ex. 9 : Find the Fourier cosine integral representation of the function $f(x) = e^{-ax}$, $x > 0$ and hence, show that

$$\int_0^\infty \frac{\cos \omega s}{1+\omega^2} d\omega = \frac{\pi}{2} e^{-x}, \quad x \geq 0.$$

Sol. : Fourier cosine integral representation of $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x \int_0^\infty f(s) \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \int_0^\infty e^{-ax} \cos \omega x \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \left[\frac{e^{-ax}}{a^2 + \omega^2} (-a \cos \omega x + s \sin \omega s) \right]_0^\infty d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \left[\frac{a}{a^2 + \omega^2} \right] d\omega$$

$$= \frac{2a}{\pi} \int_0^\infty \frac{\cos \omega x}{a^2 + \omega^2} d\omega$$

For deduction put $a = 1$,

$$\therefore \quad e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{\cos \omega x}{1+\omega^2} d\omega$$

$$\therefore \quad \int_0^\infty \frac{\cos \omega x}{1+\omega^2} d\omega = \frac{\pi}{2} e^{-x}.$$

Ex. 10 : Find Fourier cosine integral representation of the function

$$f(x) = \begin{cases} \cos x, & |x| < (\pi/2) \\ 0, & |x| > (\pi/2) \end{cases}$$

Sol. : Fourier cosine integral representation of $f(x)$ is

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x \int_0^\infty f(s) \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \int_0^{\pi/2} \cos s \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \left(\frac{1}{2}\right) \int_0^{\pi/2} [\cos (1+\omega) s + \cos (1-\omega) s] \, d\omega \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \left(\frac{1}{2}\right) \left[\frac{\sin (1+\omega) s}{1+\omega} + \frac{\sin (1-\omega) s}{1-\omega} \right]_0^{\pi/2} ds$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \left(\frac{1}{2}\right) \left[\frac{\sin \pi (1+\omega)/2}{1+\omega} + \frac{\sin \pi (1-\omega)/2}{1-\omega} \right] d\omega$$

But $\sin\left(\frac{\pi}{2} + \frac{\pi\omega}{2}\right) = \cos \frac{\pi\omega}{2}$ and $\sin\left(\frac{\pi}{2} - \frac{\pi\omega}{2}\right) = \cos \frac{\pi\omega}{2}$.

$$\therefore \quad f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x \cdot \left(\frac{1}{2}\right) \cdot \left[\frac{\cos (\pi\omega/2)}{1+\omega} + \frac{\cos (\pi\omega/2)}{1-\omega} \right] d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \cos \omega x \left(\frac{1}{2}\right) \cdot \frac{2 \cdot \cos (\pi\omega/2)}{1-\omega^2} d\omega$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos \omega x \cdot \cos (\pi\omega/2)}{1-\omega^2} d\omega.$$

EXERCISE

(A) 1. Express the function

$$f(x) = \begin{cases} e^{kx} & \text{when } x < 0 \\ e^{-kx} & \text{when } x > 0 \end{cases}$$

as a Fourier Integral and hence, show that

$$\int_0^\infty \frac{\cos \omega x}{\omega^2 - k^2} d\omega = \frac{\pi}{2k} e^{-kx} \text{ if } x > 0, \ k > 0.$$

(**Hint :** $f(x)$ is even hence, use (2)

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x \int_0^\infty e^{-ks} \cos \omega s \, d\omega \, ds = \frac{2}{\pi} \int_0^\infty \frac{k \cos \omega x}{\omega^2 - k^2} d\omega)$$

2. Express $f(x) = \begin{cases} 0, & x < 0 \\ e^{-x}, & x \geq 0 \end{cases}$

as a Fourier integral and show that

$$\int_0^\infty \frac{\cos \omega x + \omega \sin \omega x}{1+\omega^2} d\omega = \begin{cases} 0, & x < 0 \\ \pi/2, & x = 0 \\ \pi e^{-x}, & x > 0 \end{cases}$$

(**Hint :** For the second part put $x = 0$ in the integral, then

$$f(0) = \int_0^\infty \frac{1}{1+\omega^2} d\omega = \frac{1}{\pi} \left[\tan^{-1} \omega\right]_0^\infty = \frac{\pi}{2}.)$$

3. Express $f(x) = \begin{cases} \sin x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$

as a Fourier integral and show that

$$\int_0^\infty \frac{\sin \omega x \sin \pi\omega}{1-\omega^2} d\omega = \begin{cases} \frac{\pi}{2} \sin x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$$

4. Express $f(x) = \begin{cases} \cos x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$

as a Fourier integral and show that

$$\int_0^\infty \frac{\omega \sin \pi\omega \cos \omega x}{1-\omega^2} d\omega = \begin{cases} \frac{\pi}{2} \cos x, & |x| < \pi \\ 0, & |x| > \pi \end{cases}$$

(B) **1.** Express the function

$$f(x) = e^{-x} - e^{-2x}, \quad x \geq 0$$

as Fourier sine Integral and evaluate

$$\int_0^\infty \frac{\omega \sin \omega x}{(1+\omega^2)(4+\omega^2)} d\omega. \qquad \left[\textbf{Ans. : } \frac{\pi}{6}(e^{-x} - e^{-2x})\right]$$

2. Express the function $f(x) = e^{-x}$ as Fourier sine integral $(x \geq 0)$ and show that $\int_0^\infty \frac{\cos \omega x}{1+\omega^2} d\omega = \frac{\pi}{2} \cdot e^{-x}$.

3. Express $f(x) = \frac{\pi}{2} e^{-x} \cos x$ for $x > 0$ as Fourier sine integral and show that $\int_0^\infty \frac{\omega^3 \sin \omega x}{\omega^4 + 4} d\omega = \frac{\pi}{2} e^{-x} \cos x$.

4. Express $f(x) = \begin{cases} \pi/2, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$

as sine integral and show that

$$\int_0^\infty \frac{1 - \cos \pi\omega}{\omega} \sin \omega x \, d\omega = \begin{cases} \pi/2, & 0 < x < \pi \\ 0, & x > \pi \end{cases}$$

(C) **1.** Express $e^{-x} \cos x$ as Fourier cosine Integral and show that

$$e^{-x} \cos x = \frac{2}{\pi} \int_0^\infty \frac{(\omega^2 + 2)}{\omega^4 + 4} \cos \omega x \, d\omega \qquad \textbf{(S.U. 1996)}$$

2. Express the function

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1 \\ 0 & \text{for } x > 1 \end{cases}$$

as a Fourier cosine Integral and hence, show that

$$\int_0^\infty \frac{\sin \omega \cdot \cos \omega x}{\omega} d\omega = \frac{\pi}{2} \text{ if } 0 \leq x < 1.$$

Also show that the integral is equal to $\frac{\pi}{4}$ for $x = 1$ and zero for $x > 1$.

(**Hint :** $f(x) = \frac{2}{\pi} \int_0^\infty \cos \omega x \int_0^1 1 \cdot \cos \omega s \, d\omega \, ds = \frac{2}{\pi} \int_0^\infty \frac{\cos \omega x \cdot \sin \omega}{\omega} d\omega$)

3. Express $f(x) = e^{-kx} (k > 0)$ as Fourier sine and cosine Integral and show respectively that

(i) $\int_0^\infty \frac{\omega \sin \omega x}{k^2 \omega^2} d\omega = \frac{\pi}{2} e^{-kx}$

(ii) $\int_0^\infty \frac{\cos \omega x}{k^2 + \omega^2} d\omega = \frac{\pi}{2k} e^{-kx}$

4. Fourier Transform or Complex Fourier Transform

Definition : If a function $f(x)$ is defined on $(-\infty, \infty)$, is piecewise continuous in each finite interval and is absolutely integrable in $(-\infty, \infty)$ then the integral $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) \cdot e^{isx} \, dx$ is called the **Fourier Transform of** $f(x)$ and is denoted by $F\{f(x)\}$ or $F(s)$. Thus,

$$\boxed{F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) \cdot e^{isx} \, dx}$$

Ex. 1 : Find the Fourier Transform of $f(x)$, if

$$f(x) = \begin{cases} e^{i\omega x}, & a < x < b \\ 0, & x < a, \; x > b \end{cases}$$

Sol. : By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{i\omega x} \cdot e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{i(\omega+s)x} \, dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i(\omega+s)x}}{i(\omega+s)}\right]_a^b$$

$$= \frac{1}{\sqrt{2\pi}\, i(\omega+s)} (e^{i(\omega+s)b} - e^{i(\omega+s)a}).$$

Ex. 2 : Find the Fourier Transform of $f(x) = e^{-x^2/2}$.

Sol. : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-x^2/2} \cdot e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-1/2(x-is)^2} \cdot e^{-s^2/2} \, dx$$

$$= \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-1/2(x-is)^2} \, dx$$

Now, put $\frac{1}{\sqrt{2}}(x - is) = y \quad \therefore \; dx = \sqrt{2} \cdot dy$

$$\therefore \quad F(s) = \frac{e^{-s^2/2}}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-y^2} \cdot \sqrt{2} \cdot dy$$

$$= \frac{e^{-s^2/2}}{\sqrt{\pi}} \cdot \sqrt{\pi} \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}\right]$$

$$\therefore \quad F(s) = e^{-s^2/2}.$$

Ex. 3 : Find the Fourier Transform of

$$f(x) = \begin{cases} 1/2\epsilon, & |x| \le \epsilon \\ 0, & |x| > \epsilon \end{cases}$$

Sol. : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\epsilon}^{\epsilon} \frac{1}{2\epsilon} \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi} \cdot 2\epsilon} \cdot \left[\frac{e^{isx}}{is}\right]_{-\epsilon}^{\epsilon} = \frac{1}{\sqrt{2\pi} \cdot s\epsilon} \left[\frac{e^{is\epsilon} - e^{-is\epsilon}}{2}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\sin s\epsilon}{s\epsilon}.$$

Ex. 4 : Find the Fourier Transform of $f(x) = e^{-|x|}$.

Sol. : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \cdot (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \cdot \cos s\, x\, dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} \cdot \sin sx\, dx$$

Since the first integral is even and the second is odd (and hence zero.)

$$\therefore \quad F(s) = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-|x|} \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-x} \cos sx\, dx \qquad [\because |x| = x \text{ when } x > 0]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{1+s^2} (-\cos sx + s \sin sx)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \left[0 + \frac{1}{1+s^2}\right] = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+s^2}.$$

Ex. 5 : Find the Fourier Transform of

$$f(x) = \begin{cases} k, & |x| < a \\ 0, & |x| > a \end{cases}$$

Sol. : By definition

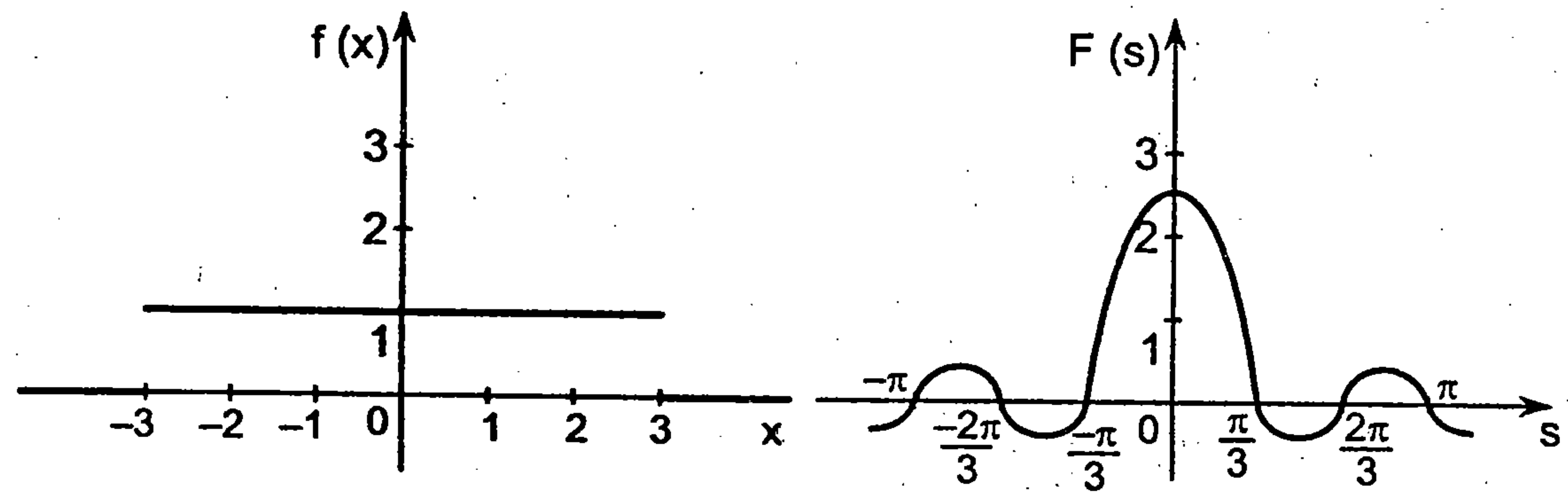
$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} k \cdot e^{isx} dx$$

$$= \frac{k}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is}\right]_{-a}^{a} = \frac{k}{\sqrt{2\pi} \cdot s} \left[\frac{e^{isa} - e^{-isa}}{i}\right]$$

$$= \frac{k \cdot 2}{\sqrt{2\pi} \cdot s} \left[\frac{e^{isa} - e^{-isa}}{2i}\right] = \frac{k}{s} \cdot \sqrt{\frac{2}{\pi}} \cdot \sin sa.$$

We show below the graph of $f(x)$ and $F(s)$ at $k = 1$ and $a = 3$.



Ex. 6 : Find the Fourier transform of

$$f(x) = \begin{cases} 1 - |x|, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

Sol. : By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|) [\cos sx + i \sin sx] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - |x|) \cos sx\, dx + \frac{i}{\sqrt{2}} \int_{-1}^{1} (1 - |x|) \sin sx\, dx$$

But the first integral is even and the second is odd hence zero.

$$\therefore \quad F(s) = \frac{2}{\sqrt{2\pi}} \int_{0}^{1} (1 - |x|) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{1} (1 - x) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[(1-x)\frac{\sin sx}{s} - \left(-\frac{\cos sx}{s^2}\right)(-1)\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(-\frac{\cos s}{s^2}\right) + \frac{1}{s^2}\right]$$

$$\therefore \quad F(s) = \sqrt{\frac{2}{\pi}} \cdot \left(\frac{1-\cos s}{s^2}\right)$$

EXERCISE

Find the Fourier Transform of the following :

1. $f(x) = \begin{cases} 1, & a < x < b \\ 0, & x < a,\ x > b \end{cases}$ $\left[\textbf{Ans. :}\ \frac{1}{\sqrt{2\pi}\cdot is}(e^{isb} - e^{isa})\right]$

2. $f(x) = e^{-x^2}$ $\left[\textbf{Ans. :}\ \frac{e^{-s^2/4}}{\sqrt{2}}\right]$

3. $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ $\left[\textbf{Ans. :}\ \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s} \cdot \sin sa\right]$

4. $f(x) = e^{-a|x|}$ $\left[\textbf{Ans. :}\ \sqrt{\frac{2}{\pi}} \cdot \frac{a}{s^2 + a^2}\right]$

(**Hint :** $f(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(a+is)x}dx + \frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-(a-is)x}$

$= \frac{1}{\sqrt{2\pi}} \cdot \left[\frac{1}{a+is} + \frac{1}{a-is}\right] = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}$)

5. $f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$

$\left[\textbf{Ans. :}\ \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s^3}\left[(a^2s^2 - 2)\sin as + 2as\cos as\right]\right]$

6. $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$ $\left[\textbf{Ans. :}\ \frac{2i}{s^2} \cdot \frac{1}{\sqrt{2\pi}}[\sin sa - as\cos sa]\right]$

7. $f(x) = \begin{cases} \frac{\pi}{2}\cos x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$ $\left[\textbf{Ans. :}\ \pi \cdot \frac{s\sin s\pi}{1-s^2}\right]$

5. Inverse Fourier Transform or Complex Fourier Transform

If $F(s)$ is the Fourier transform of $f(x)$ and if $f(x)$ satisfies certain conditions (*i.e.* Dirichlets conditions) in every finite interval $(-l, l)$ and if $\int_{-\infty}^{\infty}|f(x)|dx$ is convergent then at every point of continuity of $f(x)$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s) \cdot e^{-isx}\, ds}$$

$f(x)$ is called the **Inverse Fourier Transform** of $F(s)$.

Note : Some authors define Fourier Transforms in different ways

1. $F(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}dx$ and $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{isx}\,ds$
2. $F(s) = \int_{-\infty}^{\infty} f(x)e^{isx}dx$ and $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$
3. $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}dx$ and $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{isx}\,ds$

Even some authors use p in place of s, and $\tilde{f}(p)$ in place of $F(s)$, some use λ in place of s. We advise students to use the notation used in this book as it is more common and more convenient.

Ex. 1 : Find the Fourier transform of

$$f(x) = \begin{cases} 1, & |x| < k \\ 0, & |x| > k \end{cases} \quad \text{and hence, evaluate}$$

(i) $\int_{-\infty}^{\infty} \frac{\sin sk\cos sx}{s}\,ds$, (ii) $\int_{-\infty}^{\infty}\frac{\sin ks}{s}\,ds$,

(iii) $\int_{-\infty}^{\infty}\frac{\sin s}{s}\,ds$

Sol. : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x) \cdot e^{isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-k}^{k} 1 \cdot e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-k}^{k} = \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{s}\left[\frac{e^{isk} - e^{-isk}}{2i}\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s} \cdot \sin sk \quad \text{for } s \ne 0$$

For $s = 0$, $F(s) = \frac{1}{\sqrt{2\pi}}\int_{-k}^{k} dx = \frac{1}{\sqrt{2\pi}}[k+k] = \frac{2k}{\sqrt{2\pi}}$

Now, we use inverse Fourier Transform. We know that if

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

then, $f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} F(s) \cdot e^{-isx}\, ds$

(i) $\therefore\ f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s} \cdot \sin sk \cdot e^{-isx} \cdot ds$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(\cos sx - i \sin sx)}{s} \sin sk \cdot ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx \cdot \sin sk}{s} ds - \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\sin sx \sin sk}{s} ds$$

The second integral being odd is zero.

$$\therefore\ f(x) = \begin{cases} 1, & |x| < k \\ 0, & |x| > k \end{cases} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\cos sx \cdot \sin sk}{s} \cdot ds$$

$$\therefore\ \int_{-\infty}^{\infty} \frac{\cos sx \cdot \sin sk}{s} ds = \begin{cases} \pi, & |x| < k \\ 0, & |x| > k \end{cases}$$

(ii) In the above result, if we put $x = 0$, we put

$$\int_{-\infty}^{\infty} \frac{\sin ks}{s} ds = \pi \quad \therefore\ 2\int_{0}^{\infty} \frac{\sin ks}{s} ds = \pi$$

$$\therefore\ \int_{0}^{\infty} \frac{\sin ks}{s} ds = \frac{\pi}{2}.$$

Note : From the result (ii) above, we get

$$\boxed{\int_{0}^{\infty} \frac{\sin kx}{x} dx = \frac{\pi}{2}}$$

This is an important integral and can be used as a standard result when required. You are advised to memorise it and also the following result.

(iii) In the above result put $k = 1$,

$$\therefore\ \boxed{\int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}}$$

Ex. 2 : Find the Fourier transform of

$$f(x) = \begin{cases} (1 - x^2), & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

and hence evaluate $\displaystyle\int_{0}^{\infty} \left(\frac{x \cos x - \sin x}{x^3}\right) \cdot \cos \frac{x}{2}\, dx.$

Sol. : By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1 - x^2)\, e^{isx}\, dx$$

Integrating by parts, the integral I is given by

$$I = (1 - x^2) \cdot \frac{e^{isx}}{is} - \int \frac{e^{isx}}{is} (-2x)\, dx$$

$$= (1 - x^2) \cdot \frac{e^{isx}}{is} + \frac{2}{is} \left[x \cdot \frac{e^{isx}}{is} - \int \frac{e^{isx}}{is} \cdot 1 \cdot dx \right]$$

$$= (1 - x^2) \cdot \frac{e^{isx}}{is} + \frac{2}{is} \cdot \left[x \cdot \frac{e^{isx}}{is} - \frac{e^{isx}}{i^2 s^2} \right]$$

$$\therefore\ F(s) = \frac{1}{\sqrt{2\pi}} \left[\left(\frac{(1 - x^2) e^{isx}}{is} \right)_{-1}^{+1} + \frac{2}{is} \left(\frac{x\, e^{isx}}{is} \right)_{-1}^{+1} + \frac{2}{is} \left(\frac{e^{isx}}{s^2} \right)_{-1}^{+1} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 - \frac{4}{s^2} \left(\frac{e^{is} + e^{-is}}{2} \right) + \frac{4}{s^3} \left(\frac{e^{is} - e^{-is}}{2i} \right) \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[-\frac{4}{s^2} \cos s + \frac{4}{s^3} \sin s \right]$$

$$= -2 \cdot \sqrt{\frac{2}{\pi}} \cdot \left(\frac{s \cos s - \sin s}{s^3} \right)$$

Now, we use inverse Fourier Transform. We know that if

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx$$

then, $f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} F(s) \cdot e^{-isx}\, ds$

$$\therefore\ f(x) = \frac{1}{\sqrt{2\pi}} \left(-2 \cdot \sqrt{\frac{2}{\pi}} \right) \int_{-\infty}^{\infty} \left(\frac{s \cos s - \sin s}{s^3} \right) e^{-isx}\, ds$$

$$= -\frac{2}{\pi} \int_{-\infty}^{\infty} \cos sx \left(\frac{s \cos s - \sin s}{s^3} \right) dx$$

$$+ i \frac{2}{\pi} \int_{-\infty}^{\infty} \sin sx \left(\frac{s \cos - \sin s}{s^3} \right) ds$$

Now, the second integral being odd is zero.

$$\therefore\ f(x) = \begin{cases} (1 - x^2), & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$$= -\frac{2}{\pi} \int_{-\infty}^{\infty} \cos sx \left(\frac{s \cos s - \sin s}{s^3} \right) ds$$

Now, we put $x = \frac{1}{2}$,

$$\therefore \quad \frac{3}{4} = -\frac{2}{\pi}\int_{-\infty}^{\infty} \cos\left(\frac{s}{2}\right)\left(\frac{s\cos s - \sin s}{s^3}\right) ds$$

$$\therefore \quad \int_{-\infty}^{\infty} \cos\left(\frac{s}{2}\right)\cdot\left(\frac{s\cos s - \sin s}{s^3}\right) ds = -\frac{3\pi}{8}$$

$$\therefore \quad \int_{0}^{\infty} \cos\left(\frac{x}{2}\right)\cdot\left(\frac{x\cos x - \sin x}{x^3}\right) dx = -\frac{3\pi}{16}.$$

EXERCISE

Find the inverse Fourier Transform of $F(s) = e^{-|s|a}$.

$$\left[\textbf{Ans. :} \sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+x^2}\right]$$

(**Hint :** $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(a-ix)s}\, ds + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(a+ix)s}\, ds$

For $|s| = -s$ if $s \le 0$ and $|s| = s$ if $s \ge 0$.)

6. Fourier Sine Transform

The **Infinite Fourier Sine Transform** of $f(x)$, $0 < x < \infty$, denoted by $F_s(s)$ is defined by

$$\boxed{F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cdot \sin sx\, dx}$$

Ex. 1 : Find the Fourier sine transform of $f(x)$ if

$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & x > 1 \end{cases}$$

Sol. : By definition

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cdot\sin sx\cdot dx = \sqrt{\frac{2}{\pi}}\int_0^{\infty} 1\cdot\sin sx\cdot dx$$

$$= \sqrt{\frac{2}{\pi}}\cdot\left[-\frac{\cos sx}{s}\right]_0^1 = \sqrt{\frac{2}{\pi}}\cdot\left[\frac{1-\cos s}{s}\right]$$

Ex. 2 : Find the Fourier sine transform of $f(x)$ if

$$f(x) = \begin{cases} 0, & 0 < x < a \\ x, & a \le x \le b \\ 0, & x > b \end{cases}$$

Sol. : By definition

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cdot\sin sx\cdot dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^a 0\sin xs\cdot dx + \int_a^b x\sin sx\cdot dx + \int_b^{\infty} 0\sin sx\cdot dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\int_a^b x\sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\cdot\left[x\frac{(-\cos sx)}{s} - \int -\frac{\cos sx}{s}\cdot 1\cdot dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\cdot\left[-\frac{x\cos sx}{s} + \frac{\sin sx}{s^2}\right]_a^b$$

$$= \sqrt{\frac{2}{\pi}}\cdot\left[\frac{-b\cos bs + a\cos as}{s} + \frac{\sin bs - \sin as}{s^2}\right]$$

Ex. 3 : Find the Fourier sine transform of $f(x)$ if

$$f(x) = \begin{cases} \sin kx, & 0 \le x < a \\ 0, & x > a \end{cases}$$

Sol. : By definition

$$F_s(s) = \sqrt{\frac{2}{\pi}}\cdot\int_0^{\infty} f(x)\cdot\sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\cdot\int_0^{a} \sin kx\cdot\sin sx\, dx$$

$$= \sqrt{\frac{2}{\pi}}\cdot\int_0^{a}\frac{1}{2}\{-[\cos(k+s)x + \cos(k-s)x]\}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\left[-\frac{\sin(k+s)x}{k+s} + \frac{\sin(k-s)x}{k-s}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\left[\frac{\sin(k-s)a}{k-s} - \frac{\sin(k+s)a}{k+s}\right]$$

Ex. 4 : Find the Fourier sine transform of $f(x) = \frac{1}{x}$.

Sol. : By definition

$$F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cdot\sin sx\cdot dx \qquad \text{.......... (1)}$$

$$= \sqrt{\frac{2}{\pi}}\cdot\int_0^{\infty}\frac{1}{x}\sin sx\cdot dx$$

Put $sx = t$ $\therefore$ $s\,dx = d\,t$, when $x = 0$, $t = 0$, when $x = \infty$, $t = \infty$.

$$\therefore \quad F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} \frac{s}{t} \sin t \cdot \frac{dt}{s}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} \frac{\sin t}{t} \cdot dt$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left(\frac{\pi}{2}\right) \qquad \text{[By result (iii) of Ex. 1 of § 6]}$$

$$= \sqrt{\frac{\pi}{2}}.$$

Alternatively : To evaluate (1), consider

$$\int_0^{\infty} e^{-\alpha x} \sin sx\, dx = \frac{1}{\alpha^2 + s^2}\left[e^{-\alpha x}(-\alpha \sin sx - s \cos sx)\right]_0^{\infty}$$

$$= \frac{s}{\alpha^2 + s^2}.$$

Integrate both sides w.r.t. α between the limits α_1 and α_2,

$$\therefore \quad \int_0^{\infty}\left[\int_{\alpha_1}^{\alpha_2} e^{-\alpha x}\, d\alpha\right] \sin sx\, dx = \int_{\alpha_1}^{\alpha_2} \frac{s}{\alpha^2 + s^2} \cdot d\alpha$$

$$\therefore \quad \int_0^{\infty}\left(\frac{e^{-\alpha_1 x} - e^{-\alpha_2 x}}{x}\right) \sin sx\, dx = \tan^{-1}\frac{\alpha_2}{s} - \tan^{-1}\frac{\alpha_1}{s}$$

Now, when $\alpha_1 \to 0$ and $\alpha_2 \to \infty$,

$$\int_0^{\infty} \frac{\sin sx}{x}\, dx = \frac{\pi}{2}$$

Hence, from (1), $F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$.

For still another method see cor. of Ex. 4 of § 8.

7. Inverse Fourier Sine Transform

If $F_s(s)$ is the Fourier sine transform of $f(x)$ which satisfies Dirichlet's conditions in every finite interval $(0, l)$ and if $\int_0^{\infty} |f(x)|\, dx$ exists at every point of continuity of $f(x)$, then

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} F_s(s) \cdot \sin sx \cdot ds}$$

$f(x)$ is called the **Inverse Fourier Sine Transform** of $F_s(s)$.

Ex. 1 : Find $f(x)$ if its Fourier sine transform is e^{-as}.

Sol. : By definition

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} F_s(s) \cdot \sin sx \cdot ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} e^{-as} \cdot \sin sx\, ds$$

$$\therefore \quad f(x) = \sqrt{\frac{2}{\pi}} \cdot \left[\frac{1}{a^2 + x^2} \cdot e^{-as}(-a \sin sx - x \cos sx)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{x}{a^2 + x^2}.$$

Ex. 2 : If $F_s(s) = \frac{e^{-as}}{s}$, find $f(x)$. Hence, obtain the inverse Fourier sine transform of $\frac{1}{s}$.

Sol. : By definition

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} F_s(s) \cdot \sin sx \cdot ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} \frac{e^{-as}}{s} \sin sx\, ds \qquad \text{.......... (1)}$$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}[f(x)] = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} e^{-as} \cos sx \cdot ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{1}{a^2 + x^2} \cdot e^{-as}(-a \cos sx + x \sin sx)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + x^2}$$

Now, integrating both sides, w.r.t. x,

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1}\left(\frac{x}{a}\right) + c$$

When $x = 0$, from (1) we see that $f(0) = 0$ and $\tan^{-1} 0 = 0$ $\quad \therefore \; c = 0.$

$$\therefore \quad f(x) = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1}\left(\frac{x}{a}\right).$$

To obtain the inverse Fourier sine transform of $\frac{1}{s}$, we put $a = 0$ in the above result. We see that if

$$F_s(s) = \frac{1}{s} \text{ then } f(x) = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1} \infty = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}.$$

Ex. 3 : Find the Fourier sine transform of e^{-x}, $x \geq 0$ and hence deduce that $\int_0^\infty \frac{x \sin mx}{1+x^2} = \frac{\pi}{2} e^{-m}$, $(m > 0)$.

Sol. : By definition

$$F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty e^{-x} \cdot \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{e^{-x}}{1+s^2} (-\sin sx - s \cos sx) \right]_0^\infty = \sqrt{\frac{2}{\pi}} \cdot \frac{s}{1+s^2}$$

For deduction, we use inverse Fourier sine transform.

By definition,

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty F_s(s) \sin sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{s}{1+s^2} \cdot \sin sx \, ds$$

$$\therefore \quad e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{s \cdot \sin sx}{1+s^2} ds$$

Putting $x = m$, $\int_0^\infty \frac{s \cdot \sin sm}{1+s^2} ds = \frac{\pi}{2} e^{-m}$.

Since in definite integral variable is immaterial, changing s to x, we get

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}.$$

Remark : Fourier sine transform of $e^{-|x|}$ will be the same because in $(0, \infty)$ x is positive, *i.e.* $|x| = x$ $\therefore$ $e^{-|x|} = e^{-x}$, $x > 0$.

Ex. 4. Find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$ and hence evaluate

$$\int_0^\infty \tan^{-1}\left(\frac{x}{a}\right) \cdot \sin x \cdot dx.$$

Sol. : By definition

$$F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx \qquad \text{.......... (1)}$$

Differentiating both sides (r.h.s. under the integral sign) w.r.t. s,

$$\frac{d}{ds}[F_s(s)] = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{e^{-ax}}{x} \cos sx \cdot x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{1}{a^2+s^2} e^{-ax} (-a \cos sx + s \sin ax) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}.$$

By integration, $F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1} \frac{s}{a} + c$

But when $s = 0$, from (1) $F_s(s) = 0$ $\therefore$ $c = 0$

$$\therefore \quad F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1} \frac{s}{a} \qquad \text{.......... (2)}$$

Now, we use inverse Fourier sine transform.

By definition, $f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty F_s(s) \sin sx \, ds$

$$\therefore \quad f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \tan^{-1} \frac{s}{a} \cdot \sin sx \, ds$$

$$\therefore \quad \frac{e^{-ax}}{x} = \frac{2}{\pi} \int_0^\infty \tan^{-1} \frac{s}{a} \cdot \sin sx \cdot dx$$

Putting $x = 1$, we get

$$\frac{\pi}{2} e^{-a} = \int_0^\infty \tan^{-1} \frac{s}{a} \sin s \, ds$$

Changing s to x, we get

$$\int_0^\infty \tan^{-1} \frac{x}{a} \cdot \sin x \, dx = \frac{\pi}{2} e^{-a}.$$

Cor. : If we put $a = 0$, then from (1),

$$F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{\sin sx}{x} dx \quad \text{and}$$

from (2), $F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1} \infty = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$

$$\therefore \quad \int_0^\infty \frac{\sin sx}{x} dx = \frac{\pi}{2}.$$

8. Fourier Cosine Transform

The infinite **Fourier Cosine Transform** of $f(x)$, $0 < x < \infty$, denoted by $F_c(s)$ is defined by

$$\boxed{F_c(s) = \sqrt{\frac{2}{x}} \cdot \int_0^{\infty} f(x) \cos sx \, dx}$$

Ex. 1 : Find the Fourier cosine transform of $f(x)$ if

$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & x > 1 \end{cases}$$

Sol. : By definition,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} f(x) \cdot \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^{1} 1 \cdot \cos sx \cdot dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{\sin sx}{s}\right]_0^1 = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin s}{s}.$$

Ex. 2 : Find the Fourier cosine transform of $f(x)$ if

$$f(x) = \begin{cases} \cos kx, & 0 < x < a \\ 0, & x > a \end{cases}$$

Sol. By definition,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^{a} \cos kx \cdot \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2} \int_0^{a} [\cos (k+s) x + \cos (k-s) x] \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left[\frac{\sin (k+s) x}{k+s} + \frac{\sin (k-s) x}{k-s}\right]_0^a$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \left[\frac{\sin (k+s) a}{k+s} + \frac{\sin (k-s) a}{k-s}\right]$$

Ex. 3 : Find the Fourier cosine transform of $f(x) = e^{-x^2}$.

Sol. : By definition

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} e^{-x^2} \cdot \cos sx \, dx \qquad \text{.......... (1)}$$

Differentiating w.r.t. s, we get,

$$\frac{d}{ds}[F_c(s)] = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} -e^{-x^2} \sin sx \cdot (x) \, dx$$

Integrating by parts,

$$= -\sqrt{\frac{2}{\pi}} \cdot \left[\sin sx \cdot \left(-\frac{e^{-x^2}}{2}\right) - \int \left(-\frac{e^{-x^2}}{2}\right) \cdot (\cos sx) \, s \, dx\right]_0^{\infty}$$

$$= 0 - \frac{s}{2} \cdot \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} \cos sx \cdot e^{-x^2} dx$$

$$= -\frac{s}{2} \cdot F_c(s)$$

$$\therefore \quad \frac{d\,[F_c(s)]}{F_c(s)} = -\frac{s}{2} ds$$

By integration $\log [F_c(s)] = -\dfrac{s^2}{4} + \log c$ .... (2)

But from (1) when $s = 0$,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} e^{-x^2} dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}.$$

$$\therefore \quad \log c = \log \frac{1}{\sqrt{2}}$$

From (2), $\quad \log [F_c(s)] = -\dfrac{s^2}{4} + \log \dfrac{1}{\sqrt{2}}$

$$\therefore \quad \log \left\{\frac{F_c(s)}{1/\sqrt{2}}\right\} = -\frac{s^2}{4} \qquad \therefore \quad F_c(s) = \frac{1}{\sqrt{2}} e^{-s^2/4}.$$

9. Inverse Fourier Cosine Transform

If $F_c(s)$ is the Fourier cosine transform of $f(x)$ which satisfies Dirichlet's conditions in every finite interval $(0, l)$ and if $\int_0^{\infty} |f(x)| \, dx$ exists at every point of continuity of $f(x)$, then

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} F_c(s) \cos sx \, ds}$$

$f(x)$ is called **Inverse Fourier Cosine transform** of $F_c(s)$.

Ex. 1. Find $f(x)$ if its Fourier cosine transform is e^{-s}.

Sol. By definition,

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} e^{-s} \cos sx \cdot ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{e^{-s}}{1+x^2}(-\cos sx + x\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+x^2}.$$

Ex. 2 : Find $f(x)$ if its Fourier cosine transform is

$$F_c(s) = \begin{cases} \frac{1}{\sqrt{2x}}\left(k - \frac{s}{2}\right), & \text{if } s < 2k \\ 0, & \text{if } s > 2k \end{cases}$$

Sol. : By definition

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty F_c(s) \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^{2k} \frac{1}{\sqrt{2\pi}}\left(k - \frac{s}{2}\right) \cos sx \, ds$$

$$= \frac{1}{\pi} \cdot \left[\left(k - \frac{s}{2}\right)\frac{\sin sx}{x} - \int \frac{\sin sx}{x} \cdot \left(-\frac{1}{2}\right) ds\right]_0^{2k}$$

$$= \frac{1}{\pi} \cdot \left[\left(k - \frac{s}{2}\right) \cdot \frac{\sin sx}{x} - \frac{1}{2} \cdot \frac{\cos sx}{x^2}\right]_0^{2k}$$

$$= \frac{1}{\pi} \cdot \left[0 - \frac{1}{2x^2}\cos 2kx + \frac{1}{2x^2}\right]$$

$$= \frac{1 - \cos 2kx}{2\pi x^2} = \frac{\sin^2 kx}{\pi x^2}.$$

Ex. 3 : Find the Fourier cosine transform of $f(x)$ if

$$F_c(s) = \begin{cases} 1, & 0 < x < k \\ 0, & x \geq k \end{cases}$$

Also find $f(x)$ if $F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin ks}{s}$.

Sol. : By definition

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty f(x) \cdot \cos sx \cdot dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^k 1 \cdot \cos sx \cdot 1 \cdot dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin s k}{s}$$

Now, we use inverse Fourier cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty F_c(s) \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{\sin ks}{s} \cos sx \, ds$$

$$= \frac{2}{\pi} \cdot \left[\frac{1}{2}\int_0^\infty \frac{\sin(k+x)s + \sin(k-s)x}{s} ds\right]$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin(k+x)s}{s} ds + \frac{1}{\pi}\int_0^\infty \frac{\sin(k-x)s \, ds}{s}$$

$$\left[\because \int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2}. \text{ See note under Ex. 1 of § 6}\right]$$

$$\therefore \quad f(x) = \begin{cases} \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = 1, & \text{if } 0 < x < k \\ \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\right] = 0, & \text{if } x \geq k \end{cases}$$

(if $x > k$, then $\sin(k - x)$ becomes negative)

$$\therefore \quad f(x) = \begin{cases} 1, & \text{if } 0 < x < k \\ 0, & \text{if } x > k \end{cases}$$

Ex. 4 : Find the Fourier cosine transform of e^{-x}, $x \geq 0$ and hence deduce that $\int_0^\infty \frac{\cos mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}$.

Sol. : By definition,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty e^{-x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{e^{-x}}{1+s^2}(-\cos sx + s\sin sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{1}{1+s^2}\right]$$

Now, we use inverse Fourier cosine transform.

By definition,

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty F_c(s) \cos sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+s^2} \cos sx \, ds = \frac{2}{\pi} \cdot \int_0^\infty \frac{\cos sx}{1+s^2} ds$$

$$\therefore \quad e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\cos sx}{1+s^2} ds$$

Putting $x = m$, $\int_0^\infty \frac{\cos ms}{1+s^2} ds = \frac{\pi}{2} e^{-m}$

Since in definite integral the variable does not matter,

$$\int_0^{\infty} \frac{\cos mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m}.$$

Ex. 7 : Find the Fourier cosine transform of $f(x) = e^{-x} + e^{-2x}$, $(x > 0)$.

Sol. : By definition,

$$F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} f(x) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^{\infty} (e^{-x} + e^{-2x}) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\int_0^{\infty} e^{-x} \cos sx\, dx + \int_0^{\infty} e^{-2x} \cos sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{e^{-x}}{1+s^2}(-\cos sx + s \sin sx) + \frac{e^{-2x}}{4+s^2}(-2\cos sx + s \sin sx)\right]_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\left(0 + \frac{1}{1+s^2}\right) + \left(0 + \frac{2}{4+s^2}\right)\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{1}{1+s^2} + \frac{2}{4+s^2}\right] = \sqrt{\frac{2}{\pi}} \cdot \left[\frac{6+3s^2}{4+5s^2+s^4}\right]$$

EXERCISE

1. Find the Fourier sine and cosine transform of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

[**Ans. :** (i) $2\sqrt{\frac{2}{\pi}} \cdot \frac{\sin s}{s^2}(1-\cos s)$, (ii) $2\sqrt{\frac{2}{\pi}} \cdot \frac{\cos s}{s^2}(1-\cos s)$]

2. $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$

[**Ans. :** $\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1}\right]$;

$-\frac{1}{\sqrt{2\pi}}\left[\frac{\cos(1+s)a}{1+s} + \frac{\cos(1-s)a}{1-s} - \frac{2}{1-s^2}\right]$

3. $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x > a \end{cases}$

[**Ans. :** $-\frac{1}{\sqrt{2\pi}}\left[\frac{\cos(1+s)a}{1+s} - \frac{\cos(1-s)a}{1-s} - \frac{2}{1-s^2}\right]$;

$\frac{1}{\sqrt{2\pi}}\left[\frac{\sin(1+s)a}{1+s} + \frac{\sin(1-s)a}{1-s}\right]$

4. $f(x) = \begin{cases} k, & 0 < x < a \\ 0, & x > a \end{cases}$

[**Ans. :** (i) $k \cdot \sqrt{\frac{2}{\pi}} \cdot \left[\frac{1-\cos sa}{s}\right]$, (ii) $k \cdot \sqrt{\frac{2}{\pi}} \cdot \left(\frac{\sin sa}{s}\right)$]

5. Find Fourier Cosine Transform of $e^{-2x} + 4e^{-3x}$.

[**Ans. :** $2 \cdot \sqrt{\frac{2}{\pi}} \cdot \left[\frac{1}{s^2+4} + \frac{6}{s^2+9}\right]$]

6. Find the Fourier Sine Transform of e^{-ax}, $(a > 0)$.

[**Ans. :** $\sqrt{\frac{2}{\pi}} \cdot \frac{s}{s^2+a^2}$]

7. Find the Fourier sine transform of $e^{-|x|}$. [**Ans. :** $\sqrt{\frac{2}{\pi}} \cdot \frac{s}{1+s^2}$]

8. Find the Fourier Sine Transform of $2e^{-5x} + 5e^{-2x}$.

[**Ans. :** $s \cdot \sqrt{\frac{2}{\pi}} \cdot \left(\frac{2}{s^2+25} + \frac{5}{s^2+4}\right)$]

9. **Find Fourier sine and cosine transforms of e^{-x} and use the inversion formulae to recover the original function in both cases.**

[**Ans. :** $F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{s}{1+s^2}$, $F_c(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+s^2}$

To recover the function use Fourier integral. For $f(x) = e^{-x}$ use Fourier sine integral and Fourier Cosine Integral.]

10. Properties of Fourier Transform

1. Linearity Property : If $F(s)$ and $G(s)$ are Fourier transforms of $f(x)$ and $g(x)$, then the Fourier transform of the sum of a $f(x)$ and $bg(x)$ is given by

$$F[\,a f(x) + bg(x)\,] = aF(s) + bG(s)$$

where, a and b are constants.

Proof : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x) \cdot e \quad dx$$

7,8,9

and $G(s) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{isx} dx$

$$\therefore \quad F[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{isx} dx$$

$$= a \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x) e^{isx} dx + b \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= aF(s) + bG(s)$$

2. Change of Scale Property : If $F(s)$ is the complex Fourier transform of $f(x)$, then $F[f(ax)] = \dfrac{1}{a} F\left(\dfrac{s}{a}\right)$.

Proof : By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

where $F(s)$ denotes the Fourier transform of $f(x)$.

$$\therefore \quad F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx$$

Now put $ax = t$, $x = \dfrac{t}{a}$, $dx = \dfrac{dt}{a}$

$$\therefore \quad F[f(ax)] = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(t) \cdot e^{ist/a} \cdot \frac{dt}{a}$$

$$= \frac{1}{a} \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(t) \cdot e^{i(s/a)t} dt$$

$$\therefore \quad F[f(ax)] = \frac{1}{a} F\left(\frac{s}{a}\right).$$

3. Shifting Property : If $F(s)$ is the complex Fourier transform of $f(x)$ then $F[f(x-a)] = e^{isa} F(s)$.

Proof : By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

where $F(s)$ denotes the Fourier transform of $f(x)$.

$$\therefore \quad F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$$

Now, put $x - a = t$, $x = a + t$, $dx = dt$.

$$\therefore \quad F(x-a) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(t) e^{is(a+t)} dt$$

$$= e^{isa} \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(t) e^{ist} dt$$

$$= e^{isa} \cdot \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(x) e^{isx} dx = e^{isa} F(s).$$

4. Convolution Theorem : We know that the convolution of two functions $f(x)$ and $g(x)$ denoted by $f(x) * g(x)$ is defined by

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(x) g(x-u) du$$

Theorem : The Fourier transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of the Fourier transforms of $f(x)$ and $g(x)$.

In symbols, $F[f(x) * g(x)] = F(s) G(s)$

where, $F(s)$ and $G(s)$ denote the Fourier transforms of $f(x)$ and $g(x)$ respectively

i.e. $F(s) = F[f(x)]$ and $G(s) = F[g(x)]$

Proof : By the definition of convolution, the convolution of $f(x)$ and $g(x)$ denoted by $f(x) * g(x)$ is given by

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(u) g(x-u) du$$

(Constant is adjusted)

Taking Fourier transforms of both sides,

$$F[f(x) * g(x)] = F\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u) du\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u) du\right] e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) du \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-u) \cdot e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) du\, F[g(x-u)]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) du\, e^{ius} F[g(x)] \qquad \text{[By shifting property]}$$

$$= G(s) \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{ius} ds$$

$$= G(s) F(s)$$

Corollary : Taking the inverse Fourier transforms of both sides,

$$F^{-1}[F[f(x) * g(x)]] = F^{-1}[F(s) G(s)]$$

$$\therefore \quad f(x) * g(x) = F^{-1}[F(s) G(s)]$$

$$\therefore \quad F^{-1}[F(s)]\, F^{-1}[G(s)] = F^{-1}[F(s) * G(s)]$$

Relationship Between Fourier Transform And Laplace Transform

Let us consider the function,

$$f(t) = \begin{cases} e^{-xt} g(t), & \text{for } t > 0 \\ 0, & \text{for } t < 0 \end{cases}$$

Now, the Fourier transform of $f(t)$ is given by

$$F[f(t)] = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} f(t) e^{ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{0}^{\infty} f(t) \cdot e^{ist}\, dt \qquad [\because f(t) = 0 \text{ for } t < 0]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{0}^{\infty} e^{-xt}\, g(t) \cdot e^{ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{0}^{\infty} e^{(is-x)t} \cdot g(t)\, dt$$

Put $x - is = p$,

$$\therefore \quad F[f(t)] = \frac{1}{\sqrt{2\pi}} \cdot \int_{0}^{\infty} e^{-pt}\, g(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \cdot L[g(t)]$$

$\therefore$ Fourier transform of $f(t)$

$$= \frac{1}{\sqrt{2\pi}} \text{ Laplace transform of } g(t)$$

where $g(t)$ is defined as above.

11. Finite Fourier Sine Transform and its Inverse

(i) If $f(x)$ is a sectionally continuous function of x over a finite interval $(0, l)$ then the **finite Fourier sine transform of $f(x)$ on $(0, l)$** is defined as

$$\boxed{F_s(s) = \int_0^l f(x) \cdot \sin \frac{s\pi x}{l} \cdot dx}$$

where s is positive integer ($s = 1, 2, 3, \ldots..$)

If by proper choice of origin and scale the interval becomes $(0, \pi)$, then the **finite Fourier sine transform of $f(x)$ on $(0, \pi)$ is defined by**

$$\boxed{F_s(s) = \int_0^{\pi} f(x) \cdot \sin sx \cdot dx}$$

where s is an integer.

Note : If for finite Fourier sine transform the interval is not given, we assume it to be $(0, \pi)$.

(ii) Inverse Finite Fourier Sine Transform

If $F_s(s)$ is the finite Fourier sine transform of $f(x)$ over the interval $(0, l)$ then the inverse of $F_s(s)$ *i.e.* $f(x)$ is given by

$$\boxed{f(x) = \frac{2}{l} \sum_{s=1}^{\infty} F_s(s) \sin \frac{s\pi x}{l}}$$

If the interval is $(0, \pi)$, then

$$\boxed{f(x) = \frac{2}{\pi} \cdot \sum_{s=1}^{\infty} F_s(s) \sin sx}$$

These formulae are called **Inverse Finite Fourier Sine transforms** of $F_s(s)$.

These formulae can be proved by using Fourier Series.

Ex. 1 : Find the finite Fourier sine transform of $\sin kx$, if k is not an integer and $0 < x < \pi$.

Sol. : By definition

$$F_s(s) = \int_0^{\pi} f(x) \sin sx\, dx$$

$$= \int_0^{\pi} \sin kx \cdot \sin sx \cdot dx$$

$$= -\frac{1}{2} \int_0^{\pi} [\cos(k+s)x - \cos(k-s)x]\, dx$$

$$= -\frac{1}{2} \left[\frac{\sin(k+s)x}{k+s} - \frac{\sin(k-s)x}{k-s} \right]_0^{\pi}$$

$$= -\frac{1}{2} \cdot \left[\frac{\sin(k+s)\pi}{k+s} - \frac{\sin(k-s)x}{k-s} \right]$$

Ex. 2 : Find the finite Fourier sine transform of $\sin kx$, if k is an integer and $0 < x < \pi$.

Sol. : By definition

$$F_s(s) = \int_0^{\pi} f(x) \sin sx\, dx = \int_0^{\pi} \sin kx \cdot \sin sx\, dx.$$

Then as above

$$= -\frac{1}{2} \cdot \left[\frac{\sin(k+s)\pi}{k+s} - \frac{\sin(k-s)\pi}{k-s} \right]$$

Since s is a positive integer and k is given to be an integer, $k + s$ and $k - s$ are both integers. If $k \neq s$ and since $\sin m\pi = 0$ whenever m is an integer,

$$\therefore \quad F_s(s) = 0$$

If $k = s$, then we have

$$F_s(s) = \int_0^{\pi} \sin^2 sx\, dx = \frac{1}{2}\int_0^{\pi}(1-\cos 2sx)\, dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2sx}{2s}\right]_0^{\pi}$$

$$= \frac{\pi}{2} \quad [\because \ s \text{ is an integer}]$$

Ex. 3 : Find the finite Fourier sine transform of $f(x) = 2x$, $0 < x < 4$.

Sol. : By definition,

$$F_s(s) = \int_0^l f(x) \cdot \frac{\sin \pi x}{l} \cdot dx$$

$$= \int_0^4 2x \cdot \sin \frac{s\pi x}{4} \cdot dx \quad [\because l = 4]$$

$$= \left[2x\left(-\cos\left(\frac{s\pi x}{4}\right)\right) \cdot \frac{4}{s\pi}\right]_0^4 - \int_0^4 -\cos\left(\frac{s\pi x}{4}\right) \cdot \frac{4}{s\pi} \cdot 2 \cdot dx$$

$$= \left[-\frac{8x}{s\pi} \cdot \cos\left(\frac{s\pi x}{4}\right)\right]_0^4 + \left[\frac{8}{s\pi} \cdot \sin\left(\frac{s\pi x}{4}\right) \cdot \frac{4}{s\pi}\right]_0^4$$

$$= -\frac{32}{s\pi}\cos s\pi \qquad [\because \ s \text{ is an integer}]$$

12. Finite Fourier Cosine Transform and Its Inverse

(i) If $f(x)$ is sectionally continuous function of x over a finite interval $(0, l)$ then the **finite Fourier cosine transform of $f(x)$ on $(0, l)$** is defined as

$$\boxed{F_c(s) = \int_0^l f(x) \cdot \cos \frac{s\pi x}{l} \cdot dx}$$

where, s is a positive integer or zero ($s = 0, 1, 2, 3, ...$)

If as before by proper choice of origin and scale the interval becomes $(0, \pi)$ then the **finite Fourier cosine transform of $f(x)$ on $(0, \pi)$** is defined as

$$\boxed{F_c(s) = \int_0^{\pi} f(x) \cdot \cos sx \cdot dx}$$

where s is an integer.

Note : If for finite Fourier cosine transform interval is not given we assume it to be $(0, \pi)$.

(ii) Inverse Finite Cosine Transform

If $F_c(s)$ is the finite Fourier cosine transform of $f(x)$ over the interval $(0, l)$ then the inverse of $F_c(s)$ *i.e.* $f(x)$ is given by

$$\boxed{f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{s=1}^{\infty} F_c(s)\cos\frac{s\pi x}{l}}$$

where $F_c(0) = \int_0^l f(x)\, dx$

If the interval is $(0, \pi)$, then

$$\boxed{f(x) = \frac{1}{\pi}F_c(0) + \frac{2}{\pi}\sum_{s=1}^{\infty} F_c(s)\cos sx}$$

where $F_c(0) = \int_0^{\pi} f(x)\, dx$

These formulae are called **inverse finite Fourier transforms** of $F_s(s)$.

These formulae can be proved by using Fourier Series.

Ex. 1 : Find the finite Fourier cosine transform for $\sin kx$, $0 < x < \pi$.

Sol. : By definition,

$$F_c(s) = \int_0^{\pi} f(x)\cos sx\, dx = \int_0^{\pi} \sin kx \cdot \cos sx\, dx$$

$$= \frac{1}{2}\int_0^{\pi} [\sin(k+s)x + \sin(k-s)x]\, dx$$

$$= \frac{1}{2}\left[-\frac{\cos(k+s)x}{k+s} - \frac{\cos(k-s)x}{k-s}\right]_0^{\pi}$$

$$= \frac{1}{2}\left[-\frac{\cos(k+s)\pi}{k+s} - \frac{\cos(k-s)\pi}{k-s} + \frac{1}{k+s} + \frac{1}{k-s}\right]$$

(If k is not an integer)

If k is an integer, we see that since s is an integer.

Case (i) : if $k - s$ is even, then $k + s$ is also even,

$$\therefore \quad F_c(s) = \frac{1}{2}\left[-\frac{1}{k+s} - \frac{1}{k+s} + \frac{1}{k+s} + \frac{1}{k+s}\right] = 0$$

Case (ii) : if $k - s$ is odd, then $k + s$ is also odd,

$$\therefore \quad F_c(s) = \frac{1}{2}\left[\frac{1}{k+s} + \frac{1}{k-s} + \frac{1}{k+s} + \frac{1}{k-s}\right] = \frac{2k}{k^2 - s^2}.$$

Ex. 2 : Find the finite Fourier cosine transform for

$f(x) = 2x, \quad 0 < x < 4.$

Sol. : By definition,

$$F_c(s) = \int_0^l f(x) \cdot \cos\frac{s\pi x}{l}\, dx$$

$$= \int_0^4 2x \cdot \cos\frac{s\pi x}{4} \cdot dx \quad [\because l = 4]$$

$$= \left[2x \sin\left(\frac{s\pi x}{4}\right) \cdot \frac{4}{s\pi}\right]_0^4 - \int_0^4 \sin\left(\frac{s\pi x}{4}\right) \cdot \frac{4}{s\pi} \cdot 2\,dx$$

$$= \left[\frac{8x}{s\pi} \cdot \sin\left(\frac{s\pi x}{4}\right)\right]_0^4 + \frac{8}{s\pi}\left[\cos\left(\frac{s\pi x}{4}\right) \cdot \frac{4}{s\pi}\right]_0^4$$

$$= \frac{32}{s^2\pi^2}[\cos s\pi - 1]$$

$$= \frac{32}{s^2\pi^2}\left[(-1)^s - 1\right] \qquad [\because s \text{ is an integer}]$$

If $s = 0$, $\quad F_c(s) = \int_0^4 2x\,dx = \left[x^2\right]_0^4 = 16.$

EXERCISE

1. Find the finite Fourier sine and cosine transform of

1. $f(x) = 1$ [**Ans. : (i)** $\frac{1-(-1)^s}{s}$, **(ii)** 0 and π if $s = 0$]

2. $f(x) = x$ [**Ans. : (i)** $-\pi(-1)^s$, **(ii)** $\frac{(-1)^s - 1}{s^2}$ and $\frac{\pi^2}{2}$ if $s = 0$]

3. $f(x) = x^2$ **(S.U. 1995)** [**Ans. : (i)** $\frac{2}{s^3}(\cos s\pi - 1) - \frac{\pi^2}{s}\cos s\pi$

(ii) $\frac{2\pi}{s^3}\cos s\pi$ and $\frac{\pi^3}{3}$ if $s = 0$)

4. $f(x) = \frac{\pi}{3} - x + \frac{x^2}{2\pi}$

[**Ans. : (i)** $\frac{\pi}{6s}[(-1)^s + 2] + \frac{1}{\pi s^3}[(-1)^s - 1]$, **(ii)** $\frac{1}{s^2}$, if $s = 0$]

2. Find the finite Fourier sine transform of

1. $f(x) = \frac{x}{\pi}$ [**Ans. :** $-\frac{1}{s}(-1)^s$]

2. $f(x) = 1 - \frac{x}{\pi}$ [**Ans. :** $\frac{1}{s}$]

3. $f(x) = \frac{x}{4\pi}$ [**Ans. :** $(-1)^{s+1} \cdot \frac{1}{4s}$]

4. $f(x) = x^3$ [**Ans. :** $(-1)^s \cdot \pi\left(\frac{6}{s^3} - \frac{\pi^2}{s}\right)$]

5. $f(x) = e^{ax}$ [**Ans. :** $\frac{s}{a^2+s^2}[1-(-1)^s]e^{a\pi}$]

6. $f(x) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & (\pi/2) \le x < \pi \end{cases}$ [**Ans. :** $\frac{2}{s^2}\sin\frac{s\pi}{2}$]

3. Find the finite Fourier cosine transform of

1. $f(x) = 1 - \frac{x}{\pi}$ [**Ans. :** $\frac{1}{\pi s^2}[1-(-1)^s]$ and $\frac{\pi}{2}$, if $s = 0$]

2. $f(x) = \frac{x}{4\pi}$ [**Ans. :** $\frac{1}{4\pi s^2}[(-1)^s - 1]$ and $\frac{\pi}{8}$, if $s = 0$]

3. $f(x) = \left(1 - \frac{x}{\pi}\right)^2$ [**Ans. :** $\frac{2}{\pi s^2}$ and $\frac{\pi}{3}$ if $s = 0$]

4. $f(x) = \begin{cases} 1, & 0 < x < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < x < \pi \end{cases}$ [**Ans. :** $\frac{2}{s}\sin\frac{s\pi}{2}$, π if $s = 0$]

5. $f(x) = 3x^2$ [**Ans. :** $6\pi\frac{(-1)^s}{s^2}$, π^3 if $s = 0$]

6. $f(x) = \frac{x^2}{2\pi} - \frac{\pi}{6}$ [**Ans. :** $\frac{1}{s^2}(-1)^s$, 0 if $s = 0$]

7. $f(x) = \cos kx$ if k is not an integer and $0 < x < \pi$.

[**Ans. :** $\frac{1}{2}\left[\frac{\sin(k+s)\pi}{k+s} + \frac{\sin(k-s)\pi}{k-s}\right]$

4. 1. Find the finite Fourier sine transform of $f(x) = 2x^2$, $0 < x < c$.

[**Ans. :** $2\left[-\frac{c^3}{s\pi}\cos s\pi + \frac{2c^3}{s^3\pi^3}\cos s\pi - \frac{2c^3}{s^3\pi^3}\right]$

2. Find the finite Fourier sine transform of $f(x) = 2x$, $0 < x < 4$.

[**Ans. :** $-\frac{32}{s\pi}\cos s\pi$]

Examples On Inverse Fourier Finite Sine And Cosine Transform

We have already noted the formulae for inverse Fourier finite sine and cosine transforms. We shall see below how to apply them to find $f(x)$ when $f_s(s)$ or $f_c(s)$ is given.

Ex. 1 : Find $f(x)$ if its Fourier finite cosine transform is given by

$$F_c(s) = \frac{\left(\sin\frac{s\pi}{2} - \cos s\pi\right)}{(2s+1)\pi} \quad \text{if } s = 1, 2, 3, \ldots..$$

$$= \frac{2}{\pi}, \quad \text{if } s = 0 \qquad \text{where } 0 < x < 6.$$

Sol. : By definition, we have

$$f(x) = \frac{1}{l} F_c(0) + \frac{2}{l} \sum_{s=1}^{\infty} f_c(s) \cos \frac{s\pi x}{l}$$

Putting $l = 6$ in the above formula,

$$f(x) = \frac{1}{6} \cdot \frac{2}{\pi} + \frac{2}{6} \sum_{s=1}^{\infty} \frac{\left(\sin \frac{s\pi}{2} - \cos s\pi\right)}{(2s+1)\pi} \cos \frac{s\pi x}{6}$$

$$= \frac{1}{3\pi} + \frac{1}{3} \sum_{s=1}^{\infty} \frac{\left(\sin \frac{s\pi}{2} - \cos s\pi\right)}{(2s+1)\pi} \cos \frac{s\pi x}{6}.$$

Ex. 2 : Find $f(x)$ if its Fourier finite sine transform is given by

$$F_s(s) = \frac{2\pi(-1)^{s-1}}{s^2}, \qquad s = 1, 2, 3, \ldots.. \text{ where } 0 < x < \pi.$$

Sol. : By definition, we have,

$$F_s(s) = \frac{2}{\pi} \sum_{s=1}^{\infty} F_s(s) \sin sx$$

$$\therefore \quad F_s(s) = \frac{2}{\pi} \sum_{s=1}^{\infty} \frac{2\pi(-1)^{s-1}}{s^2} \sin sx$$

EXERCISE

1. Find $f(x)$ if its Fourier finite sine transform is

$$F_s(s) = \frac{1 - \cos s\pi}{s^2\pi^2}, \quad 0 < x < \pi$$

[**Ans. :** $f(x) = \frac{2}{\pi} \sum_{s=1}^{\infty} \left(\frac{1 - \cos s\pi}{s^2\pi^2}\right) \sin sx.$]

2. Find $f(x)$ if its Fourier finite cosine transform is

$$F_c(s) = \frac{\sin(s\pi/2)}{2s}, \qquad s = 1, 2, 3, \ldots..$$

$$= \frac{\pi}{4}, \qquad s = 0 \qquad \text{where } 0 < x < 2\pi.$$

[**Ans. :** $\frac{1}{8} + \frac{1}{\pi} \sum \frac{\sin(s\pi/2)}{2s} \cos \frac{sx}{2}.$]

EXERCISE

Theory

1. Define the following terms :
   1. Fourier Integral

   2. Fourier Cosine Integral
   3. Fourier Sine Integral
   4. Fourier Transform
   5. Inverse Fourier Transform
   6. Fourier Sine Transform
   7. Inverse Fourier Sine Transforms
   8. Fourier Cosine Transform
   9. Inverse Fourier Cosine Transform
2. State the following properties of Fourier Transform.
   1. Linearity property
   2. Change of scale property
   3. Shifting property
3. State the convolution theorem of Fourier Transform.
4. State the relationship between the Fourier Transform and Laplace Transform.
5. State finite Fourier sine transform and its inverse.
6. State finite Fourier cosine transform and its inverse.

❀ ❀ ❀

3

FOURIER SERIES

1. Introduction

The series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_1 \cos 2x + b_2 \sin 2x)$$
$$+ ..... + (a_n \cos nx + b_n \sin nx) + ... \text{ or briefly}$$

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, all a's and b's are constants is called a **trigonometric series.**

The function $f(x)$ represented by the above series is a periodic function of period 2π since, on changing x to $2\pi + x$, we get the same series.

Any periodic function $f(x)$ of period 2π which satisfies certain conditions known as Dirichlet's conditions can be expressed in the form of the above series valid for all values of x in any interval c to $c + 2\pi$ of length 2π. The expansion of $f(x)$ in the form of the above series is called **Fourier Series.**

2. Dirichlet's Conditions

A function $f(x)$ defined in the interval $c_1 \le x \le c_2$ can be expressed as Fourier Series if in the interval,

(i) $f(x)$ and its integrals are finite and single valued,

(ii) $f(x)$ has discontinuities finite in number,

(iii) $f(x)$ has finite number of maxima and minima.

These conditions are known as **Dirichlet's conditions,**

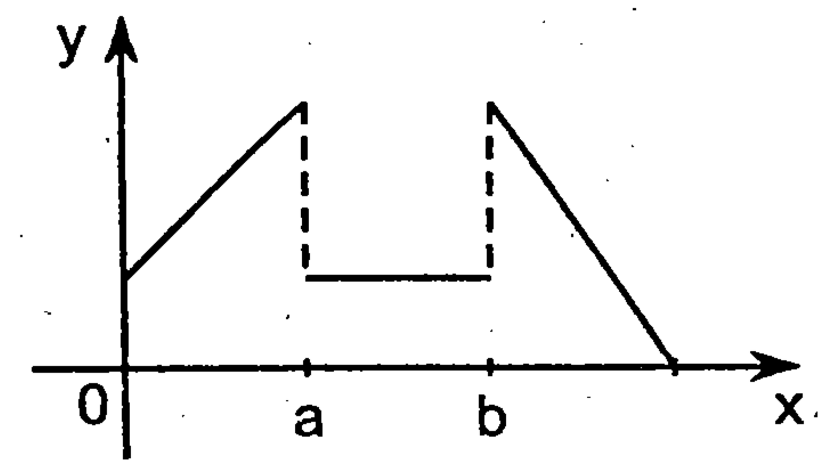
For example, $\sin^{-1} x$ cannot be expressed as Fourier Series since, it is not a single valued function.

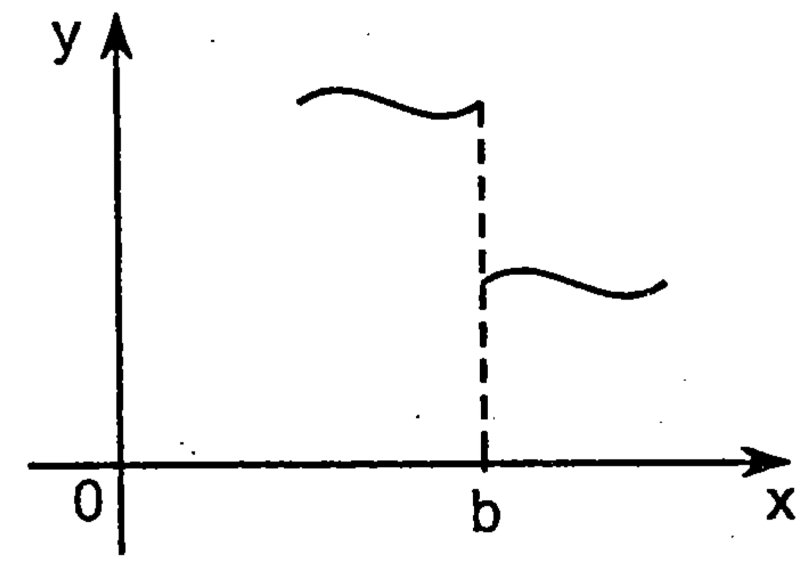
Also $\tan x$ cannot be expressed as Fourier Series in $(0, 2\pi)$ since, $\tan x$ is infinite at $x = \frac{\pi}{2}\left(\text{and}\ \frac{3\pi}{2}\right)$ which is a point on interval.

But $\cos \alpha x$ can be expanded as a Fourier Series since, it satisfies the above conditions.

(a) Functions with discontinuities

A function which has finite discontinuities can be expanded as a Fourier Series. The graph of a function as shown in the adjoining figure may consist of a finite number of disjointed curves given by different equations.





If $x = c$ is a point of finite discontinuity both the limits of $f(x)$ as $x \to c^-$ and as $x \to c^+$ exist. At such a point Fourier Series gives the value of $f(x)$ at $x = c$ as the arithmetic mean of these two limits

i.e. $$f(c) = \frac{1}{2}\left[\lim_{x \to c^-} f(x) + \lim_{x \to c^+} f(x)\right]$$

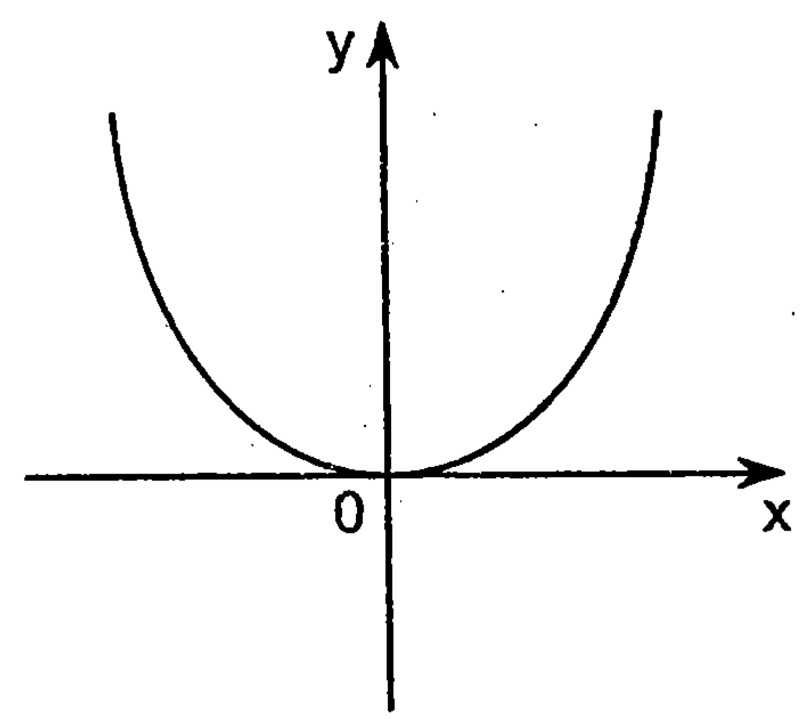
Note 1 : A function $f(x)$ is said to be **periodic** with period a if $f(a + x) = f(a)$ *e.g.* $\sin x$, $\cos x$ are periodic with period 2π.

Note 2 (a) : A function $f(x)$ is said to be **even** if $f(-x) = f(x)$ *e.g.* $\cos x$, $\sec x$, x^2 are even functions.

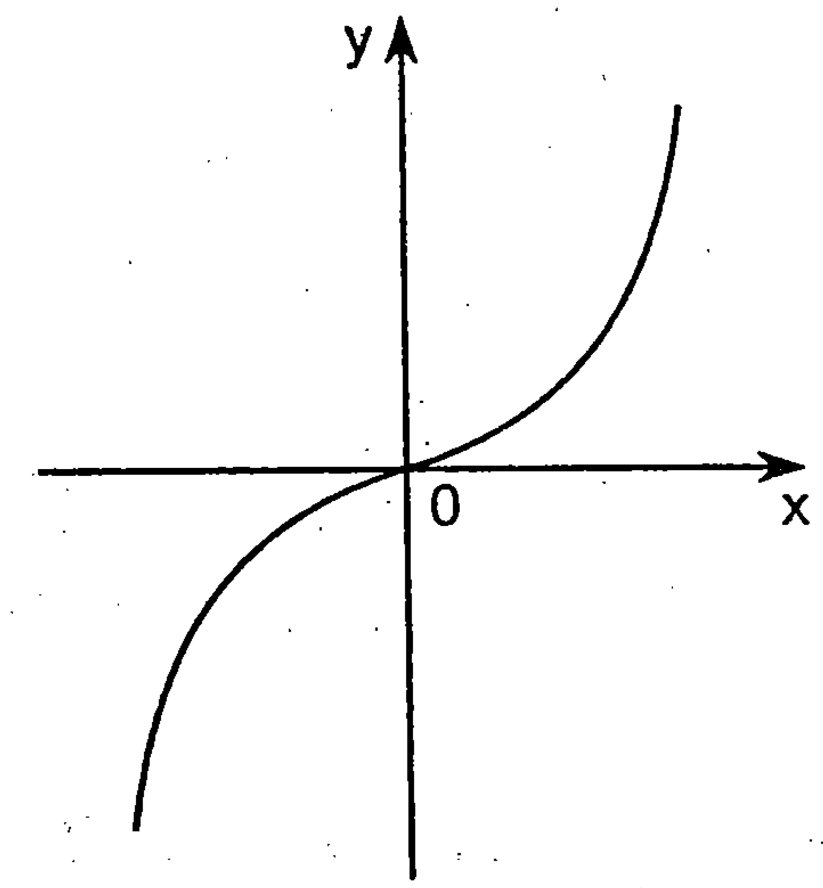
Graphically an even function is **symmetrical about the *y*-axis.**

(b) A function $f(x)$ is said to be **odd** if $f(-x) = -f(x)$ *e.g.* $\sin x$, $\operatorname{cosec} x$, x^3 are odd functions.

Geometrically an odd function is **symmetrical about the origin.**



Even Function



Odd Function

(c) A function can be neither even nor odd *e.g.* e^x, 10^x, $x^2 - x$, are neither odd nor even.

Note 3 : We also need the following results, where n is an integer

(i) $\sin n\pi = 0$ (ii) $\sin 2n\pi = 0$ (iii) $\cos n\pi = (-1)^n$ (iv) $\cos 2n\pi = 1$

(v) $\cos (n \pm 1)\pi = \cos n\pi \cos \pi \mp \sin n\pi \sin \pi = -\cos n\pi$

(vi) $\sin (n \pm 1)\pi = \sin n\pi \cos \pi \pm \cos n\pi \sin \pi = 0.$

(vii) $\sin (2n\pi + x) = \sin x$

(viii) $\cos (2n\pi + x) = \cos x$

Note 4 : We also need the following results

$2 \sin A \cos B = \sin (A+B) + \sin (A-B)$

$2 \cos A \sin B = \sin (A+B) - \sin (A-B)$

$2 \cos A \cos B = \cos (A+B) + \cos (A-B)$

$-2 \sin A \sin B = \cos (A+B) - \cos (A-B).$

Note 5 : If $\phi(x) = f(x) \cdot g(x)$ then $\phi(x)$ will be even or odd according to the following table

$f(x)$	E	E	0	0
$g(x)$	E	0	E	0
$\phi(x) = f(x) \cdot g(x)$	E	0	0	E

Note 6 : We need the following theorem

Theorem
$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad \text{if } f(x) \text{ is even.}$$
$$= 0 \quad \text{if } f(x) \text{ is odd.}$$

Note 7 : We also need the following two integrals often in Fourier series.

1. $$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$$

2. $$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx - b \sin bx]$$

Note 8 : We also need the following results

1. $$\int_c^{c+2\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0 \qquad (n \neq 0)$$

2. $$\int_c^{c+2\pi} \sin nx\,dx = \left[-\frac{\cos nx}{n}\right]_c^{c+2\pi} = 0 \qquad (n \neq 0)$$

3. $$\int_c^{c+2\pi} \sin mx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} [\sin (m+n)x + \sin (m-n)x]\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos (m+n)x}{m+n} - \frac{\cos (m-n)x}{m-n}\right]_c^{c+2\pi} = 0 \qquad (m \neq n)$$

If $m = n$,

$$\int_c^{c+2\pi} \sin nx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} 2 \sin nx \cos nx\,dx$$

$$= \frac{1}{2}\int_c^{c+2\pi} \sin 2nx\,dx = \frac{1}{2}\left[-\frac{\cos 2nx}{2n}\right]_c^{c+2\pi} = 0$$

Hence, $$\boxed{\int_c^{c+2\pi} \sin mx \cos nx\,dx = 0 \text{ for all } m, n}$$

4. $$\int_c^{c+2\pi} \cos mx \cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi} [\cos (m+n)x + \cos (m-n)x]\,dx$$

$$= \frac{1}{2}\left[\frac{\sin (m+n)x}{m+n} + \frac{\sin (m-n)x}{(m-n)}\right]_c^{c+2\pi} = 0 \qquad (m \neq n)$$

If $m = n$

$$\int_c^{c+2\pi} \cos mx \cos nx\,dx = \int_c^{c+2\pi} \cos^2 nx\,dx$$

$$= \int_c^{c+2\pi} \left(\frac{1+\cos 2nx}{2}\right) dx = \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_c^{c+2\pi} = \pi$$

Hence, $$\boxed{\int_c^{c+2\pi} \cos mx \cos nx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}}$$

5. $$\int_c^{c+2\pi} \sin mx \sin nx\,dx = -\frac{1}{2}\int_c^{c+2\pi} [\cos (m+n)x - \cos (m-n)x]\,dx$$

$$= -\frac{1}{2}\left[\frac{\sin (m+n)x}{m+n} - \frac{\sin (m-n)x}{m-n}\right]_c^{c+2\pi} = 0 \qquad (m \pm n)$$

If $m = n$

$$\int_c^{c+2\pi} \sin mx \sin nx\,dx = \int_c^{c+2\pi} \sin^2 nx\,dx$$

$$= \int_c^{c+2\pi} \left(\frac{1-\cos 2nx}{2}\right) dx = \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_c^{c+2\pi} = \pi$$

Hence, $$\boxed{\int_c^{c+2\pi} \sin mx \sin nx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}}$$

Note 9 : Putting $c = -\pi$, we get the following results.

(A) $\int_{-\pi}^{\pi} \cos nx\,dx = 0, \quad \int_{-\pi}^{\pi} \sin nx\,dx = 0$

(B) $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$ for all m, n

(c) $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0$ if $m \neq n$

$= \pi$ if $m = n$

(D) $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0$ if $m \neq n$

$= \pi$ if $m = n$

3. Determination of Fourier Coefficients (Euler's Formulae)

Let $f(x)$ be a periodic function of period 2π which can be represented in the interval $(c, c+2\pi)$ by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{.......... (1)}$$

We assume that the above series is uniformly convergent and can be integrated term by term in the given interval.

Integrating both sides of (1) from c to $c + 2\pi$, we get

$$\int_c^{c+2\pi} f(x)\,dx = \int_c^{c+2\pi} a_0\,dx + \int_c^{c+2\pi} \sum a_n \cos nx\,dx + \int_c^{c+2\pi} \sum b_n \sin nx\,dx$$

$$= a_0 [x]_c^{c+2\pi} + 0 + 0 = a_0(c + 2\pi - c) = 2a_0\pi$$

$$\therefore \boxed{a_0 = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,dx} \qquad \text{.......... (2)}$$

Now, multiply (1) by $\cos nx$ and integrate from c to $c + 2\pi$.

$$\therefore \int_c^{c+2\pi} f(x)\cos nx\,dx = a_0 \int_c^{c+2\pi} \cos nx\,dx$$

$$+ \int_c^{c+2\pi} (\Sigma a_n \cos nx) \cos nx\,dx + \int_c^{c+2\pi} (\Sigma b_n \sin x)\cos nx\,dx$$

$$= 0 + a_n\pi + 0$$

$$\therefore \boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx} \qquad \text{.......... (3)}$$

Now, multiply (1) by $\sin nx$ and integrate from c to $c + 2\pi$.

$$\therefore \int_c^{c+2\pi} f(x)\sin nx\,dx = a_0 \int_c^{c+2\pi} \sin nx\,dx$$

$$+ \int_c^{c+2\pi} (\Sigma a_n \cos nx) \sin nx\,dx + \int_c^{c+2\pi} (\Sigma b_n \sin nx)\sin nx\,dx$$

$$= 0 + 0 + b_n\pi$$

$$\therefore \boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx} \qquad \text{.......... (4)}$$

Thus, we have the following results

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \end{aligned}}$$

Cor 1 : If $c = 0$ *i.e.* if the interval is $(0, 2\pi)$, then the Fourier series of $f(x)$ is given by

$f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, where

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}}$$

Cor. 2 : If $c = -\pi$ *i.e.* if the interval is $(-\pi, \pi)$, then the Fourier series is given by

$f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, where

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}}$$

4. Parseval's Identity In (c, c + 2l)

If $f(x)$ converges uniformly in $(c, c+2l)$, then

$$\boxed{\frac{1}{2l}\int_c^{c+2l} [f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)} \qquad \text{.......... (A)}$$

This is known as **Parseval's Identity** for the function $f(x)$ in the interval $(c, c+2l)$.

Proof : We know that the Fourier Series for $f(x)$ in $(c, c+2l)$ is given by (See page 7-51).

$$f(x) = a_0 + \Sigma a_n \frac{\cos n\pi x}{l} + \Sigma b_n \frac{\sin n\pi x}{l} \qquad \text{.......... (1)}$$

Now, multiplying both sides of (1) by $f(x)$ and integrating term by term from c to $c + 2l$, we get,

$$\int_c^{c+2l} [f(x)]^2 dx = a_0 \int_c^{c+2l} f(x)\, dx + \int_c^{c+2l} \Sigma a_n f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$+ \int_c^{c+2l} \Sigma b_n f(x) \sin\left(\frac{n\pi x}{l}\right) dx \qquad \text{.......... (2)}$$

But $\int_c^{c+2l} f(x)\, dx = 2l\, a_0$ (See page 387)

$$\int_c^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx = a_n l$$

$$\int_c^{c+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx = b_n l$$

Putting these values in (2), we get,

$$\int_c^{c+2l} [f(x)]^2 dx = 2l\, a_0^2 + \Sigma a_n^2 l + \Sigma b_n^2 l$$

$$\therefore \quad \frac{1}{2l} \int_c^{c+2l} [f(x)]^2 dx = a_0^2 + \frac{1}{2} \Sigma \left(a_n^2 + b_n^2\right)$$

Cor 1 : If $l = \pi$ *i.e.* if the interval is $(c, c + 2\pi)$ and

$f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, then

$$\frac{1}{2\pi} \int_c^{c+2\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

(This is Parseval's identity for $f(x)$ in $(c, c + 2\pi)$. We have obtained this result independently below.)

Cor 2 : If $c = 0$ in the above corollary 1 *i.e.* the interval is $(0, 2\pi)$

and $f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$

then $$\frac{1}{2\pi} \int_0^{2\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

Cor 3 : If $c = -\pi$ in the above corollary 1 *i.e.* if the interval is $(-\pi, \pi)$

and $f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, then

then $$\frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

Cor 4 : If $c = 0$ in the above identity (A) *i.e.* if the interval is $(0, 2l)$

and $f(x) = a_0 + \Sigma a_n \cos \frac{n\pi x}{l} + \Sigma b_n \sin \frac{n\pi x}{l}$

then $$\frac{1}{2l} \int_0^{2l} [f(x)]^2 dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

Cor. 5 : If $c = -l$ in the above identity (A) *i.e.* if the interval is $(-l, l)$

and $f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right) + \Sigma b_n \sin\left(\frac{n\pi x}{l}\right)$

then $$\frac{1}{2l} \int_{-l}^{l} [f(x)]^2 dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)$$

(You are advised to derive this result independently.)

Cor. 6 : If the half range cosine series in $(0, \pi)$ for $f(x)$ is

$f(x) = a_0 + \Sigma a_n \cos nx$

then $$\frac{1}{\pi} \int_0^{\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots \infty\right]$$

Cor. 7 : If the half range sine series in $(0, \pi)$ for $f(x)$ is

$f(x) = \Sigma b_n \sin nx$

then $$\frac{1}{\pi} \int_0^{\pi} [f(x)]^2 dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots\ldots \infty\right]$$

Cor. 8 : If the half range cosine series in $(0, l)$ for $f(x)$ is

$$f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right)$$

then $$\frac{1}{l} \int_0^{l} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots \infty\right]$$

(We shall derive this result independently below.)

Cor. 9 : If the half range sine series in $(0, l)$ for $f(x)$ is

$$f(x) = \Sigma b_n \sin \frac{n\pi x}{c}$$

then $$\frac{1}{l} \int_0^{l} [f(x)]^2 dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots\ldots \infty\right]$$

(a) Parseval's Identity in $(c, c + 2\pi)$

If $f(x)$ converges uniformly in $(c, c + 2\pi)$, then

$$\boxed{\frac{1}{2\pi} \int_c^{c+2\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right)}$$

This is known as Parseval's Identity for the function $f(x)$ in the interval $(c, c + 2\pi)$.

Proof : We know that the Fourier series for $f(x)$ in $(c, c + 2\pi)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos x + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{..........(1)}$$

Now, multiplying both sides of (1) by $f(x)$ and integrating term by term from c to $c+2\pi$, we get,

$$\int_c^{c+2\pi} [f(x)]^2 dx = a_0 \int_c^{c+2\pi} f(x)\,dx + \int_c^{c+2\pi} \Sigma a_n f(x) \cos nx\,dx$$
$$+ \int_c^{c+2\pi} \Sigma b_n f(x) \sin nx\,dx \qquad \text{..........(2)}$$

By (2) But $\int_c^{c+2\pi} f(x)\,dx = 2\pi a_0$ [By (2), page 7-5]

By (3) $\int_c^{c+2\pi} f(x) \cos nx\,dx = a_n \pi$ [By (3), page 7-5]

By (4) $\int_c^{c+2\pi} f(x) \sin nx\,dx = b_n \pi$ [(By (4), page 7-6]

Putting these values is (2), we get,

$$\int_c^{c+2\pi} [f(x)]^2 dx = 2\pi a_0^2 + \Sigma a_n^2 \pi + \Sigma b_n^2 \pi$$

$$\therefore \quad \frac{1}{2\pi}\int_c^{c+2\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum\left(a_n^2 + b_n^2\right)$$

Cor. : Putting $c = -\pi$, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum(a_n^2 + b_n^2).$$

(b) Parseval's Identity for half range cosine series in $(0, l)$

If $f(x)$ can be expanded as a half range cosine series in $(0, l)$ as

$$f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right)$$

then $$\frac{1}{l}\int_0^l [f(x)]^2 dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots..\infty\right]$$

Proof : We know that the half range cosine series for $f(x)$ in the interval $(0, l)$ is given by (See page 7-75).

$$f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right) \qquad \text{(See § 7, page 7-75)}$$

Now. multiply both sides by $f(x)$ and integrate term by term from 0 to l.

$$\therefore \quad \int_0^l [f(x)]^2 dx = a_0 \int_0^l [f(x)]\,dx + \int_0^l \Sigma a_n f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

But $\int_0^l f(x)\,dx = a_0 l$ [Page 7-75]

$$\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx = \frac{a_n l}{2} \qquad \text{[Page 7-75]}$$

$$\therefore \quad \int_0^l [f(x)]^2 dx = a_0^2 l + \frac{1}{2}\Sigma a_n^2 l$$

$$\therefore \quad \frac{1}{l}\int_0^l [f(x)]^2 dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots..\infty\right]$$

Cor. : Putting $l = \pi$

$$\frac{1}{\pi}\int_0^{\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2}[a_1^2 + a_2^2 + a_3^2 + \ldots..\infty]$$

5. Generalised Rule Of Integration By Parts

If u, v are two functions (dashes denote the derivatives and suffixes denote the integrals), then

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots..$$

In words the rule states that, to find the integral of the product : Integral = first term × integral of he second – then write the derivative of the first term × integral of the second and repeat the procedure taking alternatively positive and negative signs. The rule is highly useful especially when the first term is a positive integral power of x and the second term can be easily integrated.

(a) Fourier Series in $(0, 2\pi)$

Ex. 1 : Find a Fourier Series to represent $f(x) = x^2$ in $(0, 2\pi)$ and hence, deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.. \qquad \textbf{(M.U. 2004)}$$

Sol. : Let $x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ in $(0, 2\pi)$ ....(1)

Then, $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\,dx$$

(By the generalised rule of integration by parts)

$$a_n = \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right] = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{4\pi^2}{n} + \frac{2}{n^3}\right\} - \left\{\frac{2}{n^3}\right\}\right] = \frac{4\pi}{n}$$

Putting these values in (1),

$$x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$\therefore\quad x^2 = \frac{4\pi^2}{3} + 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\right]$$
$$- 4\pi\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right]$$

Now, put $x = \pi$,

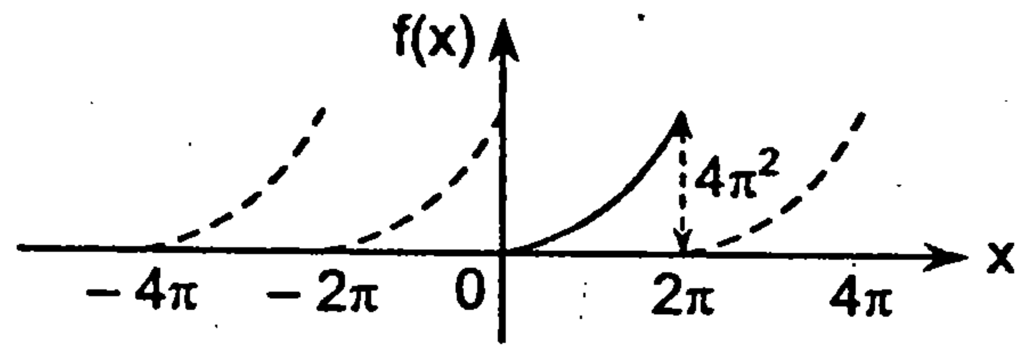
$$\therefore\quad \pi^2 = \frac{4\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\right]$$

$$-\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

$$\therefore\quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

Now, the given function, $y = x^2$ is a parabola with vertex at the origin and opening upwards and when $x = 0$, $y = 0$ when $x = 2\pi$, $y = 4\pi^2$.

The graph of the above function is shown in the following figure.



Ex. 2 : Find the Fourier Series of the function $f(x) = e^{-x}$, $0 < x < 2\pi$ and $f(x + 2\pi) = f(x)$. **(M.U. 2005)**

Hence, deduce the value of $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$

Further derive the series for cosec $h\,\pi$.

Sol. : Let $\quad f(x) = e^{-x} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots\ldots\ldots (1)$

$$\therefore\quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-x}dx = \frac{1}{2\pi}\left[-e^{-x}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left(-e^{-2\pi} + 1\right) = \frac{1 - e^{-2\pi}}{2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[e^{-x}(-\cos nx + n\sin nx)\right]_0^{2\pi} \quad \text{(See note 7, page 7-3)}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-\cos 2n\pi + n\sin 2n\pi) - e^0(-\cos 0 + n\sin 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\cdot\left[e^{-2\pi}(-1) - (-1)\right] = \frac{1 - e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[e^{-x}(-\sin nx - n\cos nx\right]_0^{2\pi} \quad \text{(See note 7, page 7-3)}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-\sin 2n\pi - n\cos 2n\pi) - e^0(-\sin 0 - n\cos 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-n) - 1(-n)\right] = \frac{n}{\pi(1+n^2)}(1 - e^{-2\pi})$$

Putting the values of a_0, a_n, b_n in (1), we get

$$e^{-x} = \frac{1 - e^{-2\pi}}{2\pi} + \frac{(1 - e^{-2\pi})}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{1+n^2}\cos nx + \sum_{n=1}^{\infty}\frac{n}{1+n^2}\sin nx\right]$$

$$= \frac{1 - e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{1+n^2}(\cos nx + n\sin nx)\right] \qquad \ldots\ldots\ldots (2)$$

Putting $x = \pi$ in (2), we get,

$$e^{-\pi} = \frac{1 - e^{-2\pi}}{2\pi} + \frac{(1 - e^{-2\pi})}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{1+n^2}\cos n\pi\right] \qquad (\because\ \sin n\pi = 0)$$

$$= \frac{1 - e^{-2\pi}}{2\pi} + \frac{(1 - e^{-2\pi})}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{1}{1+n^2}(-1)^n\right]$$

$$\therefore \quad e^{-\pi} = \frac{1-e^{-2\pi}}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} \qquad \therefore \quad \frac{\pi}{e^{\pi}(1-e^{-2\pi})} = \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$\therefore \quad \frac{\pi}{e^{\pi}-e^{-\pi}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2} \qquad \therefore \quad \frac{\pi}{2} \cdot \frac{2}{e^{\pi}-e^{-\pi}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$\therefore \quad \frac{\pi}{2} \cdot \frac{1}{\sin h\, \pi} = \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$\therefore \quad \operatorname{cosec} h\, \pi = \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n}{1+n^2}$$

$$= \frac{2}{\pi}\left[\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \ldots\right]$$

Ex. 3 : Obtain the Fourier expansion of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 \le x \le 2\pi$ and $f(x+2\pi) = f(x)$. **(M.U. 1999, 2002, 04)**

Also deduce that

(i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ (ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$ **(M.U. 2003)**

(iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$

Sol. : Let $\quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad$ .... (1)

$$\therefore \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4}\, dx$$

$$\therefore \quad a_0 = \frac{1}{8\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = -\frac{1}{24\pi}[-\pi^3 - \pi^3] = \frac{\pi^2}{12} \qquad \text{.......... (A)}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4} \cos nx\, dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

(By generalised rule of integration by parts.)

$$= \frac{1}{4\pi}\left[\left(0 + 2\pi \frac{\cos 2n\pi}{n^2} - 0\right) - \left(0 - \frac{2\pi}{n^2} - 0\right)\right]$$

$$\therefore \quad a_n = \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{n^2} \qquad [\because \cos 2n\pi = 1] \qquad \text{.......... (B)}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4} \cdot \sin nx\, dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2 \cos 2n\pi}{n} + 0 + \frac{2\cos 2n\pi}{n^3}\right) - \left(-\frac{\pi^2}{n} + 0 + \frac{2}{n^3}\right)\right]$$

$$\therefore \quad b_n = \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right] = 0 \qquad \text{.......... (C)}$$

Putting these values in (1), we get

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

$$= \frac{\pi^2}{12} + \frac{1}{1^2} \cos x + \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x + \ldots \qquad \text{.......... (2)}$$

(i) Now, put $x = 0$ in (2),

$$\therefore \quad \frac{\pi^2}{4} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad \text{.......... (3)}$$

(ii) Again, put $x = \pi$ in (2).

$$\therefore \quad 0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

$$\therefore \quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \qquad \text{.......... (4)}$$

(iii) To get the last result add (3) and (4).

$$\therefore \quad \frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \ldots$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

(iv) To derive the last result we use Parseval's identity. We know that by Parseval's identity in $(0, 2\pi)$

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum(a_n^2 + b_n^2) \qquad \text{.......... (5)}$$

Now, $\displaystyle \frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^4 dx$

$$= \frac{1}{32\pi}\int_0^{2\pi}[\pi^4 - 4\pi^3 x + 6\pi^2 x^2 - 4\pi x^3 + x^4]\, dx$$

$$= \frac{1}{32\pi}\left[\pi^4 x - 2\pi^3 x^2 + 2\pi^2 x^3 - \pi x^4 + \frac{x^5}{5}\right]_0^{2\pi}$$

$$= \frac{1}{32\pi}\left[2\pi^5 - 8\pi^5 + 16\pi^5 - 16\pi^5 + \frac{32\pi^5}{5}\right]$$

$$= \frac{1}{32\pi}\cdot\frac{2\pi^5}{5} = \frac{\pi^4}{80}$$

Hence, by (5) using (A), (B) and (C)

$$\frac{\pi^4}{80} = \frac{\pi^4}{144} + \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\ldots\right)$$

$$\pi^4\left(\frac{1}{80} - \frac{1}{144}\right) = \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right)$$

$$\frac{\pi^4}{180} = \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right)$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$$

Ex. 4 : Expand $f(x) = x \sin x$ in the interval $0 \le x \le 2\pi$.

(M.U. 2001, 02, 03, 05)

Deduce that $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2 - 1} = \frac{3}{4}$.

Sol. : Let $\displaystyle f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ .... (1)

$$\therefore \quad a_0 = \int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi}\int_0^{2\pi}(x \sin x)\cdot dx$$

$$= \frac{1}{2\pi}[(x)(-\cos x) - (-\sin x)(1)]_0^{2\pi}$$

$$= \frac{1}{2\pi}[(-2\pi + 0) - (0 + 0)] = -1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\, dx \qquad \text{.... (2)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\sin x\cos nx\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\cos nx \sin x\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(n+1)x - \sin(n-1)x]\, dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(2\pi)\left\{-\frac{\cos 2(n+1)\pi}{(n+1)} + \frac{\cos 2(n-1)\pi}{(n-1)}\right\} - 0\right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2 - 1} \quad \text{if } n \ne 1$$

If $n = 1$, the above method fails.

Putting $n = 1$ in (2), we get,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\sin x\cos x\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cdot\sin 2x\, dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{\cos 4\pi}{2}\right) - 0\right] = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx\, dx \qquad \text{.......... (3)}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\sin x \sin nx\, dx = \frac{1}{2\pi}\int_0^{2\pi} x\cdot 2\sin nx \sin x\, dx$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} x[\cos(n+1)x - \cos(n-1)x]\, dx$$

$$= -\frac{1}{2\pi}\left[x\left\{\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\cos(n+1)x}{(n+1)^2} + \frac{\cos(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= -\frac{1}{2\pi}\left[-(1)\left\{-\frac{\cos 2(n+1)\pi}{(n+1)^2} + \frac{\cos 2(n-1)\pi}{(n-1)^2}\right\} + (1)\left\{-\frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right\}\right]$$

$$= -\frac{1}{2\pi}\left[\frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right]$$

$$= 0 \quad \text{if } n \neq 1$$

If $n = 1$, the above method fails.

Putting $n = 1$ in (3), we gat,

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin^2 x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{2\pi(2\pi - 0) - \left(\frac{4\pi^2}{2} + \frac{1}{4}\right)\right\} - \left(0 - \frac{1}{4}\right)\right]$$

$$= \frac{1}{2\pi}\left[2\pi^2\right] = \pi$$

Putting these values in (1),

$$x \sin x = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1}\cos nx + \pi \sin x$$

Deduction : Putting $x = 0$, we get

$$\frac{3}{4} = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

Ex. 5 : Find the Fourier expansion for $f(x) = \sqrt{1 - \cos x}$ in $(0, 2\pi)$.

Hence, deduce that $\frac{1}{2} = \sum_{1}^{\infty} \frac{1}{4n^2 - 1}$ **(M.U. 1994, 99, 2005)**

Sol. : Let $f(x) = \sqrt{1 - \cos x} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ .... (1)

Here, $f(x) = \sqrt{2} \cdot \sin(x/2)$

$$\therefore \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{2} \cdot \sin\frac{x}{2}\, dx$$

$$= \frac{1}{\sqrt{2} \cdot \pi}\left[-2\cos\frac{x}{2}\right]_0^{2\pi} = \frac{1}{\sqrt{2} \cdot \pi}\left[-2(-1 - 1)\right] = \frac{4}{\sqrt{2} \cdot \pi} = \frac{2\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2} \cdot \sin\frac{x}{2} \cdot \cos nx\, dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{1}{2} + n\right)x + \sin\left(\frac{1}{2} - n\right)x\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{1 + 2n}\cos\left(\frac{1 + 2n}{2}\right)x - \frac{2}{1 - 2n}\cos\left(\frac{1 - 2n}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[-\frac{1}{2n + 1}\cos\left(\frac{2n + 1}{2}\right)x + \frac{1}{2n - 1}\cos\left(\frac{2n - 1}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{2}{2n + 1} - \frac{2}{2n - 1}\right] = -\frac{4\sqrt{2}}{\pi(4n^2 - 1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\frac{x}{2}\sin nx\, dx$$

$$= \frac{-\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{1}{2} + n\right)x - \cos\left(\frac{1}{2} - n\right)x\right]dx$$

$$= \frac{-\sqrt{2}}{2\pi}\left[\frac{2}{1 + 2n}\sin\left(\frac{1 + 2n}{2}\right)x - \frac{2}{1 - 2n}\sin\left(\frac{1 - 2n}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{-\sqrt{2}}{2\pi}\left[\frac{1}{2n + 1}\sin\left(\frac{2n + 1}{2}\right)x - \frac{1}{2n - 1}\sin\left(\frac{2n - 1}{2}\right)x\right]_0^{2\pi}$$

$$= 0$$

$\therefore$ Putting these valued in (1)

$$\sqrt{1 - \cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cdot \cos nx$$

Putting $x = 0$, $\frac{1}{2} = \sum_{1}^{\infty} \frac{1}{4n^2 - 1}$

Ex. 6 : Find the Fourier expansion of $\cos px$ in $(0, 2\pi)$. Hence, deduce that

$$\pi \operatorname{cosec} \pi x = \frac{1}{p} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p + n} + \frac{1}{p - n}\right]$$

Also deduce that $\pi \cot 2\pi p = \dfrac{1}{2p} + p\sum_{n=1}^{\infty} \dfrac{1}{p^2 - n^2}$.

Sol. : Let $\cos px = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ .... (1)

$$\therefore \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi} \cos px\,dx$$

$$\therefore \quad a_0 = \frac{1}{2\pi}\left[\frac{\sin px}{p}\right]_0^{2\pi} = \frac{1}{2\pi}\cdot\frac{\sin 2p\pi}{p} = \frac{\sin 2p\pi}{2\pi p}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\cos px\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2\cos px\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}[\cos(p+n)x + \cos(p-n)x]\,dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(p+n)x}{p+n} + \frac{\sin(p-n)x}{p-n}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\sin 2\pi(p+n)}{p+n} + \frac{\sin 2\pi(p-n)}{p-n}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\sin 2p\pi}{p+n} + \frac{\sin 2p\pi}{p-n}\right]$$

$$\therefore \quad a_n = \frac{1}{2\pi}\cdot\sin 2p\pi\left[\frac{1}{p+n} + \frac{1}{p-n}\right] = \frac{p}{\pi}\cdot\frac{\sin 2p\pi}{p^2 - n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\cos px\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2\sin nx\cos px\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}[\sin(n+p)x + \sin(n-p)x]\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+p)x}{n+p} - \frac{\cos(n-p)x}{n-p}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2\pi(n+p)}{n+p} - \frac{\cos 2\pi(n-p)}{n-p} + \frac{1}{n+p} + \frac{1}{n-p}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2p\pi}{n+p} - \frac{\cos 2p\pi}{n-p} + \frac{1}{n+p} + \frac{1}{n-p}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1-\cos 2p\pi}{n+p} - \frac{1-\cos 2p\pi}{n-p}\right]$$

$$= \frac{1-\cos 2p\pi}{2\pi}\left[\frac{1}{n+p} + \frac{1}{n-p}\right]$$

$$\therefore \quad b_n = \frac{1-\cos 2p\pi}{2\pi}\cdot\frac{2n}{n^2-p^2} = -\frac{n}{\pi}\frac{(1-\cos 2p\pi)}{p^2-n^2}$$

Putting these values in (1), we get,

$$\cos px = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{p^2-n^2} - \frac{(1-\cos 2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{p^2-n^2} \qquad \text{........... (2)}$$

To deduce the first result, put $x = \pi$ in (2).

$$\therefore \quad \cos p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{p^2-n^2} - \frac{(1-\cos 2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n\sin n\pi}{p^2-n^2}$$

But since, n is a positive integer, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$.

$$\therefore \quad \cos p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{p^2-n^2}$$

$$= \frac{2\sin p\pi\cos p\pi}{2p\pi} + \frac{2p\sin p\pi\cos p\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{p^2-n^2}$$

$$\therefore \quad \pi\,\mathrm{cosec}\,p\pi = \frac{1}{p} + 2p\sum_{n=1}^{\infty}\frac{(-1)^n}{p^2-n^2} = \frac{1}{p} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p+n} + \frac{1}{p-n}\right]$$

To deduce the second result put $x = 2\pi$ in (2).

$$\therefore \quad \cos 2p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2n\pi}{p^2-n^2} - \frac{(1-\cos 2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n\sin 2n\pi}{p^2-n^2}$$

But, since, n is a positive integer, $\sin 2n\pi = 0$ and $\cos 2n\pi = 1$.

$$\therefore \quad \cos 2p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{1}{(p^2-n^2)}$$

$$\therefore \quad \pi \cot 2p\pi = \frac{1}{2p} + p \sum_{n=1}^{\infty} \frac{1}{(p^2 - n^2)}$$

Ex. 7 : An alternating current i after passing through the rectifier has the form

$$i = I_0 \sin x, \quad 0 < x < \pi$$
$$= 0 \quad , \quad \pi < x < 2\pi$$

where I_0 is the maximum current and the period is 2π. Express i as a Fourier Series. Also graph the function.

Sol. : Let $\quad i = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad$ .... (1)

$$\therefore \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_0^{\pi} I_0 \sin x\,dx + \int_{\pi}^{2\pi} 0 \cdot dx\right]$$

$$= \frac{I_0}{2\pi}\left\{[-\cos x]_0^{\pi} + [0]_{\pi}^{2\pi}\right\} = \frac{I_0}{2\pi}[-\cos\pi + \cos 0]$$

$$\therefore \quad a_0 = \frac{I_0}{2\pi}[1+1] = \frac{I_0}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \quad \text{.......... (2)}$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} I_0 \sin x \cos nx\,dx + \int_{\pi}^{2\pi} 0 \cdot \cos nx\,dx\right]$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} 2\cos nx \sin x\,dx = \frac{I_0}{2\pi}\int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{I_0}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{I_0}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{I_0}{2\pi}\left[\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{I_0}{2\pi}(1+\cos n\pi)\left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{I_0}{2\pi}(1+\cos n\pi) \cdot \frac{(-2)}{n^2-1} \quad \text{when } n \neq 1$$

$$= -\frac{I_0}{\pi(n^2-1)}(1+\cos n\pi)$$

$$\therefore \quad a_n = \begin{cases} 0 & \text{when } n \text{ is odd} \\ -\dfrac{2I_0}{\pi(n^2-1)} & \text{when } n \text{ is even} \end{cases}$$

When $n = 1$, the above method fails, but from (2), we get,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} I_0 \sin x \cos x\,dx + \int_0^{2\pi} 0 \cdot \cos x\,dx\right]$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} 2\sin x\cos x\,dx = \frac{I_0}{2\pi}\int_0^{\pi}\sin 2x\,dx$$

$$\therefore \quad a_1 = \frac{I_0}{2\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = -\frac{I_0}{4\pi}[1-1] = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \quad \text{.......... (3)}$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} I_0 \sin x \sin nx\,dx + \int_0^{2\pi} 0 \cdot \sin nx\,dx\right]$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} 2\sin nx \sin x\,dx$$

$$= -\frac{I_0}{2\pi}\int_0^{\pi}[\cos(n+1)x - \cos(n-1)x]\,dx$$

$$= -\frac{I_0}{2\pi}\left[\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right]_0^{\pi}$$

$$= -\frac{I_0}{2\pi}\left[\frac{\sin(n+1)\pi}{n+1} - \frac{\sin(n-1)\pi}{n-1} - 0\right]$$

$$\therefore \quad b_n = -\frac{I_0}{2\pi}(0) \quad \text{when } n \neq 1$$

When $n = 1$, the above method fails but from (3), we get,

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} I_0 \sin x \sin x\,dx + \int_0^{\pi} 0 \cdot \sin x\,dx\right]$$

$$= \frac{I_0}{\pi}\int_0^{\pi}\sin^2 x\,dx = \frac{I_0}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx$$

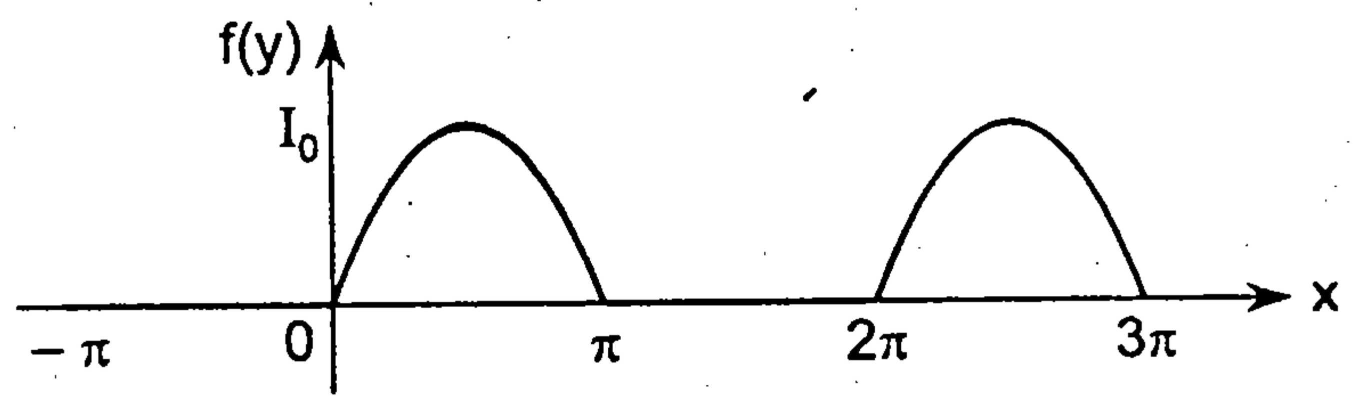
$$\therefore \quad b_1 = \frac{I_0}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{I_0}{2\pi}[\pi - 0] = \frac{I_0}{2}$$

Putting these values in (1),

$$i_0 = \frac{I_0}{\pi} + \left[0 - \frac{2I_0}{\pi(3)} \cdot \cos 2x + 0 - \frac{2I_0}{\pi(15)} \cos 4x + 0 - \frac{2I_0}{\pi(35)} \cos 6x + 0 \ldots\right] + \frac{I_0}{2} \sin x$$

$$= \frac{I_0}{\pi}\left[1 - 2\sum \frac{1}{(4n^2-1)} \cos 2nx + \frac{\pi}{2} \sin x\right]$$

The graph of the function is shown below.



Ex. 8 : Find Fourier Series for $f(x)$ in $(0, 2\pi)$.

$$f(x) = x, \quad 0 < x \leq \pi$$
$$= 2\pi - x, \quad \pi \leq x < 2\pi.$$

(M.U. 2004)

Hence, deduce that $\frac{\pi^2}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} x\,dx + \frac{1}{2\pi}\int_{\pi}^{2\pi}(2\pi - x)\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{1}{2\pi}\left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}$$

$$\therefore \quad a_0 = \frac{1}{2\pi}\left[\frac{\pi^2}{2}\right] + \frac{1}{2\pi}\left[4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right] = \frac{1}{2\pi} \cdot \pi^2 = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x \cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} + \frac{1}{\pi}\left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$\therefore \quad a_n = \frac{1}{\pi \cdot n^2}\left[(-1)^n - 1\right] - \frac{1}{\pi \cdot n^2}\left[1 - (-1)^n\right] = \frac{-2}{\pi \cdot n^2}\left[1 - (-1)^n\right]$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x \sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} + \frac{1}{\pi}\left[(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$\therefore \quad b_n = \frac{-1}{\pi n}\left[\pi(-1)^n\right] + \frac{1}{\pi n}\left[\pi(-1)^n\right] = 0$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}\cos(2n+1)\,x$$

To deduce the required result we use the Parseval's identity

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum\left(a_n^2 + b_n^2\right) \quad \ldots\ldots\ldots (1)$$

Now, $\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{1}{2\pi}\left[\int_0^{\pi} x^2 dx + \int_{\pi}^{2\pi}(2\pi - x)^2 dx\right]$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} x^2 dx + \int_{\pi}^{2\pi}(4\pi^2 - 4\pi x + x^2)\,dx\right]$$

$$= \frac{2}{2\pi}\left\{\left[\frac{x^3}{3}\right]_0^{\pi} + \left[4\pi^2 x - 2\pi x^2 + \frac{x^3}{3}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{\pi^3}{3} + 8\pi^3 - 8\pi^3 + \frac{8\pi^3}{3} - 4\pi^3 + 2\pi^3 - \frac{\pi^3}{3}\right\}$$

$$= \frac{1}{2\pi} \cdot \frac{2\pi^3}{3} = \frac{\pi^2}{3}$$

Hence, by (1), we get by using the values of a_0, a_n and b_n.

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{1}{2} \cdot \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right]$$

$$\frac{\pi^2}{12} = \frac{8}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right]$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$$

EXERCISE

Find the Fourier Series for the following functions.

1. $f(x) = e^{\alpha x}$, $(\alpha \neq 0)$ in $(0, 2\pi)$.

$$\left[\text{Ans. : } f(x) = \frac{1}{\pi}(e^{2\alpha\pi} - 1)\left[\frac{1}{2\alpha} + \sum_{n=1}^{\infty} \frac{\cos nx - n \sin nx}{\alpha^2 + n^2}\right]\right]$$

2. $f(x) = e^x$ in $(0, 2\pi)$.

$$\left[\text{Ans. : } f(x) = \frac{1}{\pi} \cdot (e^{2\pi} - 1)\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx - n \sin nx}{1 + n^2}\right]\right]$$

3. $f(x) = x^2$ in $(0, 2\pi)$.

$$\left[\text{Ans. : } f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx - 4\pi \sum_{n=1}^{\infty} \frac{1}{n} \sin nx\right]$$

4. $f(x) = x$ in $(0, 2\pi)$. **(M.U. 2003)** $\left[\text{Ans. : } f(x) = \pi - 2\sum_{n=1}^{\infty} \frac{\sin nx}{n}\right]$

5. $f(x) = \frac{1}{2}(\pi - x)$ in $(0, 2\pi)$. Hence, deduce that

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$ $\left[\text{Ans. : } f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}. \text{ Then put } x = \frac{\pi}{2}\right]$

6. $f(x) = \frac{3x^2 - 6x\pi + 2\pi^2}{12}$ in $(0, 2\pi)$. Hence, deduct that

$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ **(M.U. 2001, 04)** $\left[\text{Ans. : } \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}\right]$

7. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$

Hence, deduce that $\frac{1}{2} = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \ldots$

or $1 = \sum_{n=1}^{\infty} \frac{2}{(2n-1)(2n+1)}$.

$$\left[\text{Ans. : } f(x) = \frac{1}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos 2nx + \frac{1}{2} \sin x\right]$$

8. $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2 - \frac{\pi}{x}, & \pi \le x \le 2\pi \end{cases}$

$$\left[\text{Ans. : } f(x) = \frac{3}{4} - \frac{2}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right] + \frac{1}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right]\right]$$

9. $f(x) = \begin{cases} mx, & 0 \le x \le \pi \\ 2m\pi - mx, & \pi \le x \le 2\pi \end{cases}$

$$\left[\text{Ans. : } f(x) = \frac{m\pi}{2} - \frac{4m}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]\right]$$

10. $f(x) = \begin{cases} a, & 0 < x < \pi \\ -a, & \pi < x < 2\pi \end{cases}$

Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

$$\left[\text{Ans. : } f(x) = \frac{4a}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right]\right]$$

(For deduction put $x = \pi/2$).

11. $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$

$$\left[\text{Ans. : } f(x) = \frac{\pi}{2} - \frac{4}{5}\left[\frac{\cos x}{1} + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \ldots\right]\right]$$

12. $f(x) = \begin{cases} (x - \pi)^2, & 0 < x < \pi \\ \pi^2, & \pi < x < 2\pi \end{cases}$

$$\left[\text{Ans. : } f(x) = \frac{2}{3}\pi^2 + 2\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\right] - 2\left[\left(\pi + \frac{2}{\pi}\right)\cos x + \left(\frac{\pi}{3} + \frac{2}{9\pi}\right)\cos 3x + \ldots\right]\right]$$

13. $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$

Hence, deduce that, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

$$\left[\text{Ans. : } f(x) = \frac{3}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)\right]$$

(For deduction put $x = \pi / 2$).

14. $f(x) = \begin{cases} -\pi, & 0 < x < \pi \\ x - \pi, & \pi < x < 2\pi \end{cases}$

State the value of the series at $x = \pi$ and hence, show that

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

$$\left[\textbf{Ans. : } f(x) = -\frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right] + \left[-3\sin x - \frac{1}{2}\sin 2x - \sin 3x + \dots\right]\right.$$

$$\left. f(\pi)\frac{-\pi+0}{2} = -\frac{\pi}{2}.\ \text{Then put } x = \pi.\right]$$

15. If $f(x) = 2x$, $0 \le x \le 2\pi$, find the Fourier series of $f(x)$. Also find a_4 and b_{10}. **(M.U. 2002)**

$$\left[\textbf{Ans. : } f(x) = 2\pi - 4\sum_{n=1}^{\infty} \frac{\sin nx}{n};\ a_4 = 0,\ b_{10} = -0.4\right]$$

(b) Fourier Expansion of $f(x)$ in the interval $(-\pi, \pi)$

Ex. 1 : Obtain the Fourier expansion of e^x in $-\pi < x < \pi$.

Sol. : Let $e^x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{2\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[e^{\pi} - e^{-\pi}\right] = \frac{1}{\pi}\sinh\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx = \frac{1}{\pi}\left[\frac{1}{1+n^2}\cdot e^x(\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{1}{a^2+b^2}\cdot e^{ax}(a\cos bx + b\sin bx)\right]$$

$$\therefore\ a_n = \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi\right]$$

$$= \frac{2\cos n\pi}{\pi(1+n^2)}\cdot\frac{e^{\pi}-e^{-\pi}}{2} = \frac{2\cos n\pi}{\pi(1+n^2)}\cdot\sinh\pi = \frac{(-1)^n}{\pi(1+n^2)}\cdot 2\sinh\pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin x\,dx = \frac{1}{\pi}\left[\frac{1}{1+n^2}\cdot e^x(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$\left[\therefore \int e^{ax}\sin bx\,dx = \frac{1}{a^2+b^2}\cdot e^{ax}(a\sin bx - b\cos bx)\right]$$

$$\therefore\ b_n = \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[-e^{\pi}\cdot n\cos n\pi + ne^{-\pi}\cos n\pi\right]$$

$$= \frac{-2\cos n\pi}{\pi(1+n^2)}\left[\frac{e^{\pi}-e^{-\pi}}{2}\right] = -\frac{2n\cos n\pi}{\pi(1+n^2)}\cdot\sinh\pi$$

$$= -(-1)^n\cdot\frac{n}{\pi(1+n^2)}\cdot 2\sinh\pi$$

$$\therefore\ e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + (-1)^n\cdot\sum_{n=1}^{\infty}\frac{1}{(1+n^2)}\cos nx - (-1)^n\cdot\sum_{n=1}^{\infty}\frac{n}{(1+n^2)}\sin nx\right]$$

$$\therefore\ e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{2}\cdot\cos x + \frac{1}{5}\cdot\cos 2x - \frac{1}{10}\cdot\cos 3x + \dots + \frac{1}{2}\cdot\sin x - \frac{2}{5}\cdot\sin 2x + \frac{3}{10}\cdot\sin 3x - \dots\right]$$

Ex. 2 : Find the Fourier Series for periodic function

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

State the value of $f(x)$ at $x = 0$ and hence, deduce that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$ **(M.U. 1996, 2003)**

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ .... (1)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -\pi\cdot dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{2\pi}\left[-\pi\left\{x\right\}_{-\pi}^{0} + \left\{\frac{x^2}{2}\right\}_0^{\pi}\right] = \frac{1}{2\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi(0-0) + \left\{\pi(0) + \frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n}\right)\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\frac{(1-\cos n\pi)}{n} + \left\{\pi\left(-\frac{\cos n\pi}{n}\right) - 0\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right]$$

$$= \frac{1}{n}[1 - 2\cos n\pi] = \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Putting these values in (1),

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{1-2(-1)^n}{n}\sin nx$$

$$= -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$$

$$+\left[3\cdot\frac{\sin x}{1} - \frac{1}{2}\sin 2x + \sin 3x - \ldots\right] \qquad \text{.......... (2)}$$

Now, $f(x)$ is discontinuous at $x = 0$. At a point of discontinuity $x = c$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right]$$

$$\therefore \quad f(0) = \frac{1}{2}[-\pi + 0] = -\frac{\pi}{2}$$

Hence, putting $x = 0$ in (2),

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Ex. 3 : Find the Fourier Series for $f(x) = \begin{cases}\cos x, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi\end{cases}$.

Sol. : Let $f(x) = a_o + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ .... (1)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}\cos x\,dx + \int_{0}^{\pi}\sin x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\{\sin x\}_{-\pi}^{0} + \{-\cos x\}_{0}^{\pi}\right] = \frac{1}{2\pi}[(0) + (1+1)] = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\cos nx\,dx + \int_{0}^{\pi}\sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 2\cos nx\cos x\,dx + \int_{2}^{\pi} 2\cos nx\sin x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left[\cos(n+1)x + \cos(n-1)x\right]dx\right.$$

$$\left. + \int_{0}^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{-\pi}^{0}\right.$$

$$\left. + \left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\{0\} + \left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right\}\right] \text{ if } n \neq 1$$

$$\therefore a_n = \begin{cases} 0 & \text{if } n \text{ is odd and} \neq 1 \\ -\dfrac{2}{\pi(n^2-1)} & \text{if } n \text{ is even}\end{cases}$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\cos x\,dx + \int_{0}^{\pi}\sin x\cos x\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(\frac{1+\cos 2x}{2}\right)dx + \int_{0}^{\pi}\frac{\sin 2x}{2}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)_{-\pi}^{0} + \left(-\frac{\cos 2x}{4}\right)_{0}^{\pi}\right]$$

$$\therefore\ a_1 = \frac{1}{\pi}\left[\frac{1}{2}\left\{0-(-\pi-0)-\left(\frac{1}{4}-\frac{1}{4}\right)\right\}\right] = \frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\sin nx\,dx + \int_{0}^{\pi}\sin x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 2\cos x\sin nx\,dx + \int_{0}^{\pi} 2\sin x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}[\sin(n+1)x - \sin(n-1)x]\,dx - \int_{0}^{\pi}[\cos(n+1)x - \cos(n-1)x]\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right\}_{-\pi}^{0} - \left\{\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\left\{-\frac{1}{n+1} - \frac{1}{n-1} + \frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - 0\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1+\cos n\pi}{n+1} - \frac{1+\cos n\pi}{n-1}\right] = -\frac{n}{\pi}\frac{(1+\cos n\pi)}{n^2-1} \quad \text{if } n \neq 1$$

$$\therefore\ b_n = \begin{cases} 0 & \text{if } n \text{ is odd and} \neq 1 \\ -\dfrac{2n}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\sin x\,dx + \int_{0}^{\pi}\sin^2 x\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\frac{\sin 2x}{2}\,dx + \int_{0}^{\pi}\frac{1-\cos 2x}{2}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos 2x}{2}\right\}_{-\pi}^{0} + \frac{1}{2}\left\{x - \frac{\sin 2x}{2}\right\}_{0}^{\pi}\right]$$

$$\therefore\ b_1 = \frac{1}{\pi}\left[-\left(\frac{1}{2}-\frac{1}{2}\right) + \frac{1}{2}(\pi - 0)\right] = \frac{1}{2}$$

Putting these values in

$$f(x) = a_0 + a_1\cos x + [a_2\cos 2x + a_4\cos 4x + \ldots] + b_1\sin x + [b_2\sin 2x + b_4\sin 4x + \ldots]$$

$$= \frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \ldots\right] + \frac{1}{2}\sin x - \frac{2}{\pi}\left[\frac{2\sin 2x}{3} + \frac{4\sin 4x}{15} + \ldots\right]$$

$$= \frac{1}{\pi} + \frac{1}{2}(\cos x + \sin x) + \frac{2}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{(1-4n^2)}\cos 2nx + \sum_{n=1}^{\infty}\frac{2n}{(1-4n^2)}\sin 2nx\right]$$

EXERCISE

Find Fourier expansion of

1. $f(x) = e^{ax}$ in $(-\pi, \pi)$.

$$\left[\textbf{Ans. : } e^{ax} = \frac{\sin h\,a\pi}{a\pi} + \frac{2\sin h\,a\pi}{\pi}\sum\frac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)\right]$$

2. $f(x) = e^{-ax}$ in $(-\pi, \pi)$ and hence, show that

$$\frac{\pi}{\sin h a\pi} = \frac{1}{a} + \sum\frac{(-1)^n\cdot 2a}{n^2+a^2}$$

$$\left[\textbf{Ans. : } f(x) = \frac{2\sin h\,a\pi}{\pi}\left[\frac{1}{2a} - \frac{a\cos x}{a^2+1} + \frac{a\cos 2x}{a^2+2^2} - \ldots + \frac{\sin x}{a^2+1} - \frac{\sin 2x}{a^2+2^2} + \frac{\sin 3x}{a^2+3^2} - \ldots\right]\right]$$

3. $f(x) = \begin{cases} 1/2, & -\pi < x < 0 \\ x/\pi, & 0 < x < \pi \end{cases}$.

Hence deduce that $\dfrac{\pi^2}{8} = \sum_{n=0}^{\infty}\dfrac{1}{(2n+1)^2}$. **(M.U. 2002)**

$$\left[\textbf{Ans. : } f(x) = \frac{1}{2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin 2nx}{n}\right]$$

4. $f(x) = \begin{cases} x+\pi, & 0 \le x \le \pi \\ -x-\pi, & -\pi \le x \le 0 \end{cases}$ **(M.U. 2003)**

$$\left[\textbf{Ans. : } f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi}{(2n-1)^2} + 4\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}\right]$$

5. $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases}$ and $f(x+2\pi) = f(x)$.

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

$\dfrac{\pi}{4} = \dfrac{1}{1} - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$

$$\left[\textbf{Ans. : } f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cdot\cos(2n+1)x - \sum\frac{(-1)^n}{n}\sin nx\right]$$

6. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$

Hence, deduce that

(i) $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \ldots = \dfrac{1}{2}$

(ii) $\dfrac{1}{4}(\pi - 2) = \dfrac{1}{1\cdot 3} - \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} - \ldots$ **(M.U. 2000, 03, 04)**

$$\left[\textbf{Ans. : } f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\left[\frac{\cos 2x}{4\cdot 1^2 - 1} + \frac{\cos 4x}{4\cdot 2^2 - 1} + \ldots\right]\right]$$

7. $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$

$$\left[\textbf{Ans. : } f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right]\right.$$
$$\left. - \left[\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right]\right]$$

8. $f(x) = \begin{cases} x - \pi, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$ **(M.U. 2003)**

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

$$\left[\textbf{Ans. : } f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\sin 5x + \ldots\right]\right.$$
$$\left. + 4\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]\right]$$

(c) Even and odd functions in $(-\pi, \pi)$

(i) Even function in $(-\pi, \pi)$: If $f(x)$ is even in the interval $(-\pi, \pi)$, then,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{2\pi}\int_0^{\pi} f(x)\,dx \qquad [\because f(x) \text{ is even}]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \quad [\because f(x)\cos nx \text{ is even}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \qquad [\because f(x)\sin nx \text{ is odd}]$$

(ii) Odd function in $(-\pi, \pi)$: If $f(x)$ is odd in the interval $(-\pi, \pi)$, then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0 \qquad [\because f(x) \text{ is odd}]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \qquad [\because f(x)\cos nx \text{ is odd}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \quad [\because f(x)\sin nx \text{ is even}]$$

Thus, we have,

If $f(x)$ is even in $(-\pi, \pi)$

$$\boxed{a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad b_n = 0}$$

If $f(x)$ is odd in $(-\pi, \pi)$

$$\boxed{a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx}$$

Ex. 1 : Find the Fourier expansion of $f(x) = x^2$, $-\pi \le x \le \pi$, and hence, prove that

(i) $\dfrac{\pi^2}{6} = \displaystyle\sum_1^{\infty}\frac{1}{n^2}$, (ii) $\dfrac{\pi^2}{12} = \displaystyle\sum_1^{\infty}\frac{(-1)^{n+1}}{n^2}$,

(iii) $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

Sol. : Here $f(x)$ is an even function because $f(-x) = (-x)^2 = x^2 = f(x)$ [$\therefore b_n = 0$]

Let $f(x) = a_0 + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$ ....(1)

$$\therefore \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 - (2\pi)\left(-\frac{\cos n\pi}{n^2}\right) + 0\right\} - \left\{0\right\}\right]$$

$$= \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

Putting these values in (1)

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

(i) For first deduction put $x = \pi$.

$$\therefore \quad \pi^2 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n}{n^2}\cos n\pi$$

$$\pi - \frac{\pi^2}{3} = 4\sum\frac{(-1)^n(-1)^n}{n^2} = 4\sum\frac{(-1)^{2n}}{n^2}$$

$$\therefore \quad \frac{2\pi^2}{3}\cdot\frac{1}{4} = \sum\frac{(-1)^{2n}}{n^2} \qquad \therefore \quad \frac{\pi^2}{6} = \sum\frac{1}{n^2}$$

(ii) For second deduction put $x = 0$.

$$\therefore \quad 0 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n}{n^2}\cdot 1$$

$$\therefore \quad -\frac{\pi^2}{3} = 4\sum\frac{(-1)^n}{n^2} \qquad \therefore \quad \frac{\pi^2}{12} = (-1)\sum\frac{(-1)^n}{n^2}$$

$$\therefore \quad \frac{\pi^2}{12} = \sum\frac{(-1)^{n+1}}{n^2}$$

(iii) For third deduction, add the above two results.

$$\therefore \quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots.$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots.$$

$$\therefore \quad \frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.$$

Cor. Assuming the validity of term by term differentiation of the series for x^2, find the series for x and hence, deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.$$

Sol. : We have

$$x^2 = \frac{\pi^2}{3} + 4\sum_1^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\sin 3x + \ldots..\right]$$

Differentiating w.r.t. x

$$2x = 4\left[\frac{1}{1^2}\sin x - \frac{2}{2^2}\sin 2x + \frac{3}{3^2}\sin 3x - \ldots.\right]$$

$$\therefore \quad x = 2\left[\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots.\right]$$

Putting $x = \frac{\pi}{2}$, we get, $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \ldots.$

Ex. 2 : It is given that for $-\pi < x < \pi$,

$$x^2 = \frac{\pi^2}{3} + 4\sum(-1)^n\cdot\frac{\cos nx}{n^2}.$$

By using Parseval's identiyly prove that $\sum\frac{1}{n^4} = \frac{\pi^4}{90}$. **(M.U. 2004)**

Sol. : By Pareseval's identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum(a_n^2 + b_n^2)$$

Here, $f(x) = x^2$, $a_0^2 = \frac{\pi^2}{3}$, $a_n = \frac{4(-1)^n}{n^2}$, $b_n = 0$.

$$\therefore \quad \frac{1}{2\pi}\int_{-\pi}^{\pi}x^4 dx = \frac{1}{2\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{\pi^4}{5}$$

$$\therefore \quad \frac{\pi^4}{5} = \frac{\pi^4}{9} + \frac{1}{2}\cdot 16\left[\frac{1}{1^4} + \frac{1}{2^4} + \ldots..\right]$$

$$\therefore \quad \pi^4\left(\frac{1}{5} - \frac{1}{9}\right) = 8\left[\frac{1}{1^4} + \frac{1}{2^4} + \ldots..\right]$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots..$$

Ex. 3 : Obtain Fourier expansion of $f(x) = |\cos x|$ in $(-\pi, \pi)$.

Sol. : Here $f(x)$ is an even function because $f(-x) = |\cos(-x)| = f(x)$. $[\therefore b_n = 0]$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ .... (1)

$$\therefore \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} |\cos x|\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x \cdot dx\right]$$

$[\because |\cos x| = \cos x$, for $0 < x < \pi/2$
and $|\cos x| = -\cos x$, for $\pi/2 < x < \pi$.]

$$= \frac{1}{\pi}\left[\{\sin x\}_0^{\pi/2} - \{\sin x\}_{\pi/2}^{\pi}\right]$$

$$\therefore \quad a_0 = \frac{1}{\pi}[\{1-0\} - \{0-1\}] = \frac{2}{\pi} \quad \text{.......... (2)}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx\,dx - \int_{\pi/2}^{\pi} \cos x \cos nx\,dx\right] \quad \text{.......... (3)}$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos nx \cos x\,dx - \int_{\pi/2}^{\pi} 2\cos nx \cos x\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \{\cos(n+1)x + \cos(n-1)x\}\,dx - \int_{\pi/2}^{\pi} \{\cos(n+1)x + \cos(n-1)x\}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_0^{\pi/2} - \left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} - 0\right\} - \left\{0 - \frac{\sin(n+1)\pi/2}{n+1} - \frac{\sin(n-1)\pi/2}{n-1}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n(\pi/2)}{n+1} - \frac{\cos n(\pi/2)}{n-1}\right]$$

$$\therefore \quad a_n = \begin{cases} -\dfrac{4}{\pi(n^2-1)} \cos n\dfrac{\pi}{2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd and } n \neq 1 \end{cases} \quad \text{.......... (4)}$$

To find a_1, we put $n = 1$ in (3)

$$\therefore \quad a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x\,dx - \int_{\pi/2}^{\pi} \cos^2 x\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x + \frac{\sin 2x}{2}\right\}_0^{\pi/2} - \left\{x + \frac{\sin 2x}{2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi}{2}\right\} - \left\{\pi - \frac{\pi}{2}\right\}\right] = 0 \quad \text{.......... (5)}$$

Putting these values from (2) and (4) in (1), we get,

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \ldots\right]$$

Ex. 4 : Obtain Fourier Series for the function $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

Deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$ **(M.U. 1995, 97, 2001, 05)**

Sol. : Here $f(x)$ is an even function because

$$f(-x) = \begin{cases} 1 - \dfrac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1 + \dfrac{2x}{\pi}, & 0 \le -x \le \pi \end{cases} = \begin{cases} 1 - \dfrac{2x}{\pi}, & \pi \ge x \ge 0 \\ 1 + \dfrac{2x}{\pi}, & 0 \ge x \ge -\pi \end{cases}$$

$$= f(x) \quad [\therefore b_n = 0]$$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$$\therefore \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right) \cdot dx$$

$$= \frac{1}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = \frac{1}{\pi}[\pi - \pi] = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right)-\left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0-\frac{2\cos n\pi}{\pi n^2}\right\}-\left\{-\frac{2}{\pi n^2}\right\}\right]$$

$$= \frac{4}{\pi^2 n^2}[1-\cos n\pi]$$

$$= \begin{cases} 0\ , & \text{if } n \text{ is even} \\ \dfrac{8}{\pi^2 n^2}, & \text{if } n \text{ is odd}\end{cases}$$

$$\therefore \quad f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots..\right]$$

Now by putting $x = 0$ in the above result, we can get

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots..$$

Ex. 5 : Obtain the Fourier expansion of

$$f(x) = \begin{cases} \cos x, & -\pi < x < 0 \\ -\cos x, & 0 < x < \pi \end{cases} \text{ and } f(x+2\pi) = f(x).$$

Sol. : Here $f(x)$ is an odd function because

$$f(-x) = \begin{cases} \cos(-x), & -\pi < -x < 0 \\ -\cos(-x), & 0 < -x < \pi \end{cases} = \begin{cases} \cos x, & \pi > x > 0 \\ -\cos x, & 0 > x > -\pi \end{cases}$$

$$= -f(x) \quad [\therefore a_n = 0]$$

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ .... (1)

Now $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin nx\,dx$ .... (2)

$$= -\frac{1}{\pi}\int_0^{\pi} 2\sin nx\cos x\,dx$$

$$= -\frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x+\sin(n-1)x]\,dx$$

$$= -\frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1}-\frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1}-\frac{\cos(n-1)\pi}{n-1}+\frac{1}{n+1}+\frac{1}{n-1}\right]$$

$$= -\frac{1}{\pi}\left[\frac{1+\cos n\pi}{n+1}+\frac{1+\cos n\pi}{n-1}\right]$$

$$= -\frac{2n}{\pi}\frac{(1+\cos n\pi)}{n^2-1} \quad \text{if } n \neq 1$$

$$\therefore \quad b_n = \begin{cases} 0 & \text{if } n \text{ is odd and } \neq 1 \\ -\dfrac{4n}{\pi(n^2-1)} & \text{if } n \text{ is even}\end{cases}$$

Now putting $n = 1$ in (2)

$$b_1 = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin x\,dx = -\frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$$

$$= -\frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = 0$$

Putting these values in (1),

$$\therefore \quad f(x) = -\frac{4}{\pi}\left[\frac{2}{3}\sin 2x+\frac{4}{15}\sin 4x+\frac{6}{35}\sin 6x+\ldots..\right]$$

$$= -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{2n}{4n^2-1}\sin 2nx$$

Ex. 6 : Obtain Fourier series for

$$f(x) = x+\frac{\pi}{2}, \quad -\pi < x < 0$$

$$= \frac{\pi}{2}-x, \quad 0 < x < \pi$$

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots.$

Also deduce that $\frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots.$ **(M.U. 1999, 2002)**

Sol. : Clearly $f(x)$ is an even function

$$f(-x) = \begin{cases} -x+\dfrac{\pi}{2}, & -\pi < -x < 0 \\ \dfrac{\pi}{2}+x, & 0 < -x < \pi\end{cases} = \begin{cases} \dfrac{\pi}{2}-x, & \pi > x > 0 \\ \dfrac{\pi}{2}+x, & 0 < x < -\pi\end{cases}$$

$$= f(x) \quad [\therefore b_n = 0]$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)dx = \frac{1}{\pi}\left[\frac{\pi}{2}x-\frac{x^2}{2}\right]_0^{\pi} = 0 \quad \ldots\ldots (1)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2}-x\right)\left(\frac{\sin nx}{n}\right)-(-1)\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-(-1)^n\cdot\frac{1}{n^2}+\frac{1}{n^2}\right] = \frac{2}{\pi}\left[\frac{1}{n^2}-\frac{(-1)^n}{n^2}\right]$$

$$\therefore\quad f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\ldots\right] \qquad \ldots\ldots\ldots (2)$$

(i) Now $f(x)$ is discontinuous at $x = 0$

But at a point of discontinuity $x = c$.

$$f(x) = \frac{1}{2}\left[\lim_{x\to c^-} f(x)+\lim_{x\to c^+} f(x)\right] \qquad \therefore\ f(0) = \frac{1}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = \frac{\pi}{2}$$

$\therefore$ Putting $x = 0$,

$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right]$$

$$\therefore\quad \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$$

(ii) We now use Parseval's identity in $(-\pi, \pi)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = a_0^2+\frac{1}{2}\sum(a_n^2+b_n^2) \qquad \ldots\ldots\ldots (3)$$

$$\therefore\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(x+\frac{\pi}{2}\right)^2dx+\int_0^{\pi}\left(\frac{\pi}{2}-x\right)^2dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(x^2+\pi x+\frac{\pi^2}{4}\right)dx+\int_0^{\pi}\left(\frac{\pi^2}{4}-\pi x+x^2\right)dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{x^3}{3}+\frac{\pi x^2}{2}+\frac{\pi^2}{4}x\right\}_{-\pi}^{0}+\left\{\frac{\pi^2}{4}x-\frac{\pi x^2}{2}+\frac{x^3}{3}\right\}_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[0-\left\{-\frac{\pi^3}{3}+\frac{\pi^3}{2}-\frac{\pi^3}{4}\right\}+\left\{\frac{\pi^3}{4}-\frac{\pi^3}{2}+\frac{\pi^3}{3}\right\}-0\right]$$

$$= \frac{2\pi^3}{2\pi}\left[\frac{1}{4}-\frac{1}{2}+\frac{1}{3}\right] = \frac{\pi^2}{12}$$

Hence, from (1), (2) and (3),

$$\frac{\pi^2}{12} = \frac{1}{2}\cdot\frac{16}{\pi^2}\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right]$$

$$\frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots$$

Ex. 7 : Prove that $\sin h\,ax = \frac{2}{\pi}\sin h\,a\pi\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot n}{n^2+a^2}\sin nx\right]$

(M.U. 1998)

Sol. : Here $f(x)$ is an odd function because

$$f(-x) = \sin h\,(-ax) = \frac{e^{-ax}-e^{ax}}{2}$$

$$= -\frac{e^{ax}-e^{-ax}}{2} = -\sin h\,ax = -f(x) \qquad [\therefore a_n = 0]$$

$\therefore$ Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$\therefore\quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin h\,ax\sin nx\,dx$$

To find b_n, we use the result

$$\cos(n-ia) = \cos nx\cos i\,ax+\sin nx\sin i\,ax$$

$$= \cos nx\cos h\,ax+i\sin nx\sin h\,ax.$$

$$(\because\ \cos ix = \cos hx,\ \sin ix = i\sin hx)$$

Hence,

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin nx\sin h\,ax\,dx = \frac{2}{\pi}\text{I.P.}\int_0^{\pi}\cos(n-ia)x\,dx$$

$$= \frac{2}{\pi}\text{I.P.}\left[\frac{\sin(n-ia)x}{n-ia}\right]_0^{\pi} = \frac{2}{\pi}\text{I.P.}\frac{\sin(n-ia)\pi}{(n-ia)}$$

$$= \frac{2}{\pi}\text{I.P.}(n+ia)\frac{\sin(n-ia)\pi}{n^2-i^2a^2} = \frac{2}{\pi}\text{I.P.}\frac{(n+ia)}{n^2+a^2}\sin(n\pi-ia\pi)$$

$$= \frac{2}{\pi}\text{I.P.}\frac{(n+ia)}{(n^2+a^2)}[\sin n\pi\cos ia\pi-\cos n\pi\sin ia\pi]$$

$$= \frac{2}{\pi}\text{I.P.}\frac{(n+ia)}{n^2+a^2}[-i\cos n\pi\sin h\,a\pi]$$

$$= \frac{2}{\pi}\text{I.P.}\frac{(n+ia)}{n^2+a^2}(-i)(-1)^n\sin h\,a\pi$$

$$= \frac{2n}{(n^2+a^2)\pi}(-1)^{n+1}\sin h\,a\pi$$

Alternatively $b_n = \frac{2}{\pi}\int_0^{\pi}\frac{e^{ax}-e^{-ax}}{2}\cdot\sin nx\,dx$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+x^2}(a\sin x - n\cos nx) - \frac{e^{-ax}}{a^2+x^2}(-a\sin nx - n\cos nx)\right]_0^{\pi}$$

$$= \frac{2n}{(n^2+a^2)\pi}\cdot(-1)^{n+1}\sin ha\pi \quad \text{as before.}$$

$\therefore \quad f(x) = \sum b_n \sin nx$

$$= \frac{2}{\pi}\sin h\,a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot n}{n^2+a^2}\sin nx.$$

Ex. 8 : Obtain Fourier Series of $x\cos x$ in $(-\pi, \pi)$. **(M.U. 2003)**

Sol. : Since, $f(-x) = (-x)\cos(-x)$

$= -x\cos x = -f(x)$, $f(x)$ is an odd function $[\therefore a_n = 0]$

$\therefore$ Let $f(x) = x\cos x = \sum_{n=1}^{\infty} b_n \sin nx$

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx\,dx \qquad \text{.......... (1)}$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cdot 2\sin nx\cos x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x + \sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx$$

$$= \frac{1}{\pi}\left\{\left[x\cdot\frac{-\cos(n+1)x}{n+1} - (1)\cdot\frac{-\sin(n+1)x}{(n+1)^2}\right]_0^{\pi} + \left[x\cdot\frac{-\cos(n-1)x}{n-1} - (1)\cdot\frac{-\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n+1}\cdot\cos(n+1)\pi - \frac{\pi}{n-1}\cos(n-1)\pi\right]$$

$$= \frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1} \qquad [\because \cos(n\pi \pm \pi) = -\cos n\pi]$$

$$= (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right] = (-1)^n\cdot\frac{2n}{n^2-1} \quad \text{if } n \neq 1$$

For $n = 1$, we put $n = 1$ in (1).

$$\therefore \quad b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\cdot\frac{-\cos 2x}{2} - (1)\cdot\frac{-\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi}{2}\right)\right] = -\frac{1}{2}$$

Hence, $x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}(-1)^n\cdot\frac{n}{n^2-1}\sin nx$

Ex. 9 : Find the Fourier expansion of $f(x) = x + x^2$ when $-\pi \le x \le \pi$ and $f(x + 2\pi) = f(x)$.

Hence, deduce that **(i)** $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$

(ii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ **(M.U. 1996, 2001)**

Sol. : We first note that $f(x) = x + x^2$ is the sum of the odd function $f_1(x) = x$ and the even function $f_2(x) = x^2$.

Hence, Fourier expansion of $f(x)$ is the sum of the Fourier expansions of $f_1(x)$ and $f_2(x)$.

Now, since $f_1(x) = x$ is an odd function $[\because a_n = 0]$

Let $f_1(x) = x = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{\pi(-1)\frac{(-1)^n}{n} - 0\right\} - \left\{0\right\}\right] = \frac{2(-1)^{n+1}}{n}$$

$$\therefore \quad f_1(x) = x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin nx \qquad \text{.......... (2)}$$

Now, since $f_2(x) = x^2$ is an even function $[\because b_n = 0]$

Let $f_2(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cdot \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 + 2\pi\frac{\cos n\pi}{n^2} - 0\right\} - \left\{0\right\}\right]$$

$$= 4\,\frac{(-1)^n}{n^2}$$

$$\therefore \quad f_2(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \qquad \text{.......... (3)}$$

Form (2) and (3)

$$x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

For deduction, it is enough to consider the series of x^2. We have obtained in (3).

$$x^2 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n}{n^2}\cos nx \qquad \text{.......... (4)}$$

(i) Putting $x = \pi$ in (4)

$$\pi^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \ldots..\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots..\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots.. \qquad \text{.... (5)}$$

(ii) Putting $x = 0$ in (4)

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots..\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots.. \qquad \text{.......... (6)}$$

(iii) Adding (5) and (6)

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots..\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots..$$

Ex. 10 : Prove that

$$\frac{x}{12}(\pi - x)(2\pi - x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n^3} \quad \text{where} \quad 0 \le x \le 2\pi$$

Sol. : If we put $x = z + \pi$ then when $x = 0$, $z = -\pi$ and when $x = 2\pi$, $z = \pi$. Thus, the interval changes from 0 to 2π to $-\pi$ to π.

$$F(z) = \frac{(z+\pi)}{12}\cdot(\pi - z - \pi)\cdot(2\pi - z - \pi)$$

$$= \frac{(z+\pi)}{12}\cdot(-z)\cdot(\pi - z) = \frac{z}{12}\cdot(z^2 - \pi^2)$$

Now $F(-z) = -\frac{z}{12}\cdot(z^2 - \pi^2) = -F(z)$

$\therefore$ $F(z)$ is an odd function ($\therefore$ $a_n = 0$). Let $F(z) = \sum b_n \sin nz$

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} F(z)\sin nz\,dz = \frac{2}{\pi}\int_0^{\pi}\frac{1}{12}\cdot(z^3 - \pi^2 z)\sin nz\,dz$$

$$= \frac{1}{6\pi}\left[(z^3 - \pi^2 z)\left(-\frac{\cos nz}{n}\right) - (3z^2 - \pi^2)\left(-\frac{\sin nz}{n^2}\right) + (6z)\left(\frac{\cos nz}{n^3}\right) - 6(1)\left(\frac{\sin nz}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{1}{6\pi}\left[\left\{0 - 0 + \frac{6\pi\cos n\pi}{n^3} - 0\right\} - \left\{0\right\}\right] = \frac{\cos n\pi}{n^3} = \frac{(-1)^n}{n^3}$$

$$\therefore \quad F(z) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nz$$

Putting $z = x - \pi$, we get

$$f(x) = \frac{x}{12}(\pi - x)(2\pi - x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n(x - \pi)$$

$$= \sum\frac{(-1)^n}{n^3}(-1)\sin(n\pi - nx)$$

$$= \sum\frac{(-1)^n}{n^3}(-1)\{\sin n\pi\cos nx - \cos n\pi\sin nx\}$$

$$= \sum \frac{(-1)^n}{n^3}(-1)\left\{0-(-1)^n \sin nx\right\} = \sum_{n=1}^{\infty} \frac{1}{n^3} \sin nx.$$

EXERCISE

Find the Fourier Series for $f(x)$ where

1. $f(x) = \cos px$ $(-\pi, \pi)$, where p is not an integer. Hence, prove that

$$\cot p\pi = \frac{2p}{\pi}\left[\frac{1}{2p^2} - \frac{1}{p^2-1^2} + \frac{1}{p^2-2^2} - \frac{1}{p^2-3^2} + \ldots\right]$$

And deduce that $\cos\theta = \frac{1}{\theta} - \sum_{n=1}^{\infty} \frac{2\theta}{n^2\pi^2-\theta^2}$.

Also deduce that $\frac{1}{2} - \frac{\pi\sqrt{3}}{18} = \frac{1}{9.1^2-1} + \frac{1}{9.2^2-1} + \frac{1}{9.3^2-1} + \ldots$

(M.U. 1993, 96)

[**Ans. :** $\cos px = \frac{2p\sin p\pi}{\pi}\left[\frac{1}{2p^2} - \frac{\cos x}{p^2-1^2} + \frac{\cos 2x}{p^2-2^2} - \frac{\cos 3x}{p^2-3^2} + \ldots\right]$

Now put $x = \pi$,

$$\therefore \quad \cot p\pi = \frac{2p}{\pi}\left[\frac{1}{2p^2} + \frac{1}{p^2-1^2} + \frac{1}{p^2-2^2} + \ldots\right]$$

Now, put $p\theta = \pi$

$$\therefore \quad \cot\theta = \frac{1}{\theta} - 2\theta\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2-\theta^2}$$

Now put $p = 1/3$,

$$\frac{1}{\sqrt{3}} = \frac{2}{3\pi}\left[\frac{9}{2} + \frac{9}{1-9\cdot 1^2} + \frac{9}{1-9\cdot 2^2} + \frac{9}{1-9\cdot 3^2} + \ldots\right]$$]

2. $f(x) = |\sin x|$ in $(-\pi, \pi)$. **(M.U. 2003, 04)**

[**Ans. :** $f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots\right]$]

3. $f(x) = \sqrt{1-\cos x}$ in $(-\pi, \pi)$ and hence, deduce that $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2-1}$.

[**Ans. :** $f(x) \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum \frac{1}{4n^2-1} \cdot \cos nx$. Then put $x = 0$]

4. $f(x) = |x|$ in $(-\pi, \pi)$. Hence, deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$ **(M.U. 2004)**

(**Hint :** $f(x) = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases}$)

[**Ans. :** $\frac{\pi}{2} - \frac{4}{2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$]

5. $f(x) = \sin ax$, $-\pi < x < \pi$ **(M.U. 1997, 2004)**

[**Ans. :** $f(x) = \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \ldots\right]$]

6. $f(x) = x^3$, in $(-\pi, \pi)$ [**Ans. :** $f(x) = \sum_{n=1}^{\infty} (-1)^n \cdot \left(\frac{12}{n^3} - \frac{2\pi^2}{n}\right)\sin nx$]

7. $f(x) = x\sin x$ in $(-\pi, \pi)$. Hence, deduce that

$$\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \ldots$$ **(M.U. 2000, 02)**

[**Ans. :** $f(x) = 1 - \frac{1}{2}\cos x - 2\cdot\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2-1}\cos nx$

i.e. $f(x) = 1 - \frac{1}{2}\cos x - \frac{2}{3}\cos 2x + \frac{2}{8}\cos 3x - \frac{2}{15}\cos 4x + \ldots$

Now, put $x = \pi/2$.]

8. $f(x) = \sin px$ in $(-\pi, \pi)$.

[**Ans. :** $f(x) = \frac{2\sin p\pi}{\pi}\left(\sum_{n=1}^{\infty}(-1)^{n-1} \cdot \frac{n}{n^2-p^2} \cdot \sin nx\right)$

where p is not an integer]

9. $f(x) = \cos h\, px$, $-\pi < x < \pi$

[**Ans. :** $f(x) = \frac{2a\sin h\, p\pi}{\pi}\left[\frac{1}{2p^2} + \sum \frac{(-1)^n \cdot \cos nx}{n^2+a^2}\right]$]

(**Hint :** Refer to solved ex. 6

$a_n = \frac{2}{\pi}\text{R.P.}\int_0^{\pi} \cos(n+ia)\,x\,dx$, $b_n = 0.$)

10. $f(x) = x - x^2$, $-\pi < x < \pi$. Hence, deduce that,

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$ **(M.U. 1998, 2003)**

$$\left[\textbf{Ans. : } f(x) = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx\right]$$

11. $f(x) = \dfrac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}}, \quad -\pi < x < \pi$ **(M.U. 2003)**

(**Hint :** $f(x) = \dfrac{\sin h\, ax}{\sin h\, a\pi}$. See ex. 6) $\left[\textbf{Ans. : } f(x) = \dfrac{2}{\pi}\sum \dfrac{(-1)^{n+1}\cdot n}{n^2 + a^2}\cdot \sin nx\right]$

12. $f(x) = \dfrac{\pi^2}{12} - \dfrac{x^2}{4}$ in $(-\pi, \pi)$

$$\left[\textbf{Ans. : } f(x) = \frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots\ldots\right]$$

13. (a) $f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$ **(M.U. 2003)**

$$\left[\textbf{Ans. : } f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right]\right]$$

(b) $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$ **(M.U. 2004)**

(**Hint :** Put $x = \pi + z$ then $f(x) = \begin{cases} \pi + z, & -\pi \le z \le 0 \\ \pi - z, & 0 \le z \le \pi \end{cases}$)

$$\left[\textbf{Ans. : } \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\ldots\right)\right]$$

14. $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$ **(M.U. 2004)**

$$\left[\textbf{Ans. : } f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x\right]$$

15. $f(x) = \dfrac{x(\pi^2 - x^2)}{12}, \quad -\pi < x < \pi.$ **(M.U. 2004)**

$$\left[\textbf{Ans. : } f(x) = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots\ldots\right]$$

16. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$

Also deduce that $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \ldots\ldots = \dfrac{1}{2}$

and $\dfrac{1}{1\cdot 3} - \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} - \ldots\ldots = \dfrac{\pi - 2}{4}.$ **(M.U. 2004)**

17. $f(x) = \dfrac{x(\pi - x)(\pi + x)}{12}$ in $(-\pi, \pi)$.

Hence, find $\dfrac{1}{1^3} - \dfrac{1}{3^3} + \dfrac{1}{5^3} - \dfrac{1}{7^3} + \ldots\ldots$ **(M.U. 2004)**

$$\left[\textbf{Ans. : } f(x) = \sum \frac{(-1)^{n+1}}{n^3}\sin nx. \text{ Then put } x = \frac{\pi}{2}\right]$$

6. Fourier Series in $(c, c + 2l)$

In many engineering problems the period of expansion of a function is not 2π but say $2l$. To apply previous-theory to this interval we change the interval from c to $c + 2l$ to c to $c + 2\pi$ by changing the variable x to z as explained below.

Let $f(x)$ be defined in the interval c to $c + 2l$.

To transform the interval from $2l$ to 2π, we put $x = \dfrac{lz}{\pi}$ or $z = \dfrac{\pi x}{l}$.

Now when $x = c$, $z = \dfrac{\pi c}{l} = d$ say and when $x = c + 2l$, $z = \dfrac{\pi(c + 2l)}{l} = \dfrac{\pi c}{l} + 2\pi = d + 2\pi$.

Thus, the function $f(x)$ of period $2l$ in the interval $(c, c + 2l)$ is transformed into the function $f\left(\dfrac{lz}{\pi}\right) = F(z)$ say of period 2π in the interval $(d, d + 2\pi)$.

Hence, we can write,

$$f(x) = f\left(\frac{lz}{\pi}\right) = F(z) = a_0 + \sum a_n \cos nz + \sum b_n \sin nz \qquad \ldots\ldots\ldots (1)$$

where, $a_0 = \dfrac{1}{2\pi}\displaystyle\int_d^{d+2\pi} F(z)\,dz,$

$$a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\cos nz\,dz$$

$$b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\sin nz\,dz \qquad \ldots\ldots\ldots (2)$$

Now, in these expressions we make inverse substitution $z = \dfrac{\pi x}{l}$, $dz = \dfrac{\pi}{l}dx$, when $z = d$, $x = c$ and when $z = d + 2\pi$, $x = c + 2l$ (as seen above).

Hence, (2) gives

$$\left.\begin{aligned} a_0 &= \frac{1}{2l}\int_c^{c+2l} f(x)\,dx \\ a_n &= \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ b_n &= \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \end{aligned}\right\} \quad \text{.......... (3)}$$

And (1) becomes $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

Cor. 1 : If $c = 0$ *i.e.* if the interval is 0 to $2l$ then putting $c = 0$ in (3), the constants are given by

$$\begin{aligned} a_0 &= \frac{1}{2l}\int_0^{2l} f(x)\,dx \\ a_n &= \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ b_n &= \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \end{aligned}$$

Cor. 2 : If $c = -l$ *i.e.* if the interval is $-l$ to l, then putting $c = -l$ in (3), the constants are given by

$$\begin{aligned} a_0 &= \frac{1}{2l}\int_{-l}^{l} f(x)\,dx \\ a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \end{aligned}$$

(a) Fourier Series in the interval (0, 2l)

Ex. 1 : Find the Fourier expansion of x^2 in $(0, a)$. **(M.U. 2005)**

Hence, deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$

Sol. : Here $2l = a$ $\therefore$ $l = a/2$.

Let $x^2 = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

i.e. $x^2 = a_0 + \sum a_n \cos\frac{2n\pi x}{a} + \sum b_n \sin\frac{2n\pi x}{a}$ .... (1)

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{a}\int_0^a x^2 dx = \frac{1}{a}\left[\frac{x^3}{3}\right]_0^3 = \frac{a^2}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{a}\int_0^a x^2 \cdot \cos\frac{2n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left[x^2\left(\frac{a}{2n\pi}\cdot\sin\frac{2n\pi x}{a}\right) - (2x)\left(-\frac{a^2}{4n^2\pi^2}\cos\frac{2n\pi x}{a}\right) + (2)\left(-\frac{a^3}{8n^3\pi^3}\sin\frac{2n\pi x}{a}\right)\right]_0^a$$

$$= \frac{2}{a}\left[\left\{0 + (2a)\left(\frac{a^2}{4n^2\pi^2}\right) + 0\right\} - \left\{0\right\}\right] = \frac{a^2}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{a}\int_0^a x^2 \cdot \sin\frac{2n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left[x^2\left(-\frac{a}{2n\pi}\cdot\cos\frac{2n\pi x}{a}\right) - (2x)\left(-\frac{a^2}{4n^2\pi^2}\cdot\sin\frac{2n\pi x}{a}\right) + 2\left(\frac{a^3}{8n^3\pi^3}\cdot\cos\frac{2n\pi x}{a}\right)\right]_0^a$$

$$= \frac{2}{a}\left[\left\{a^2\left(-\frac{a}{2n\pi}\right) - 0 + 2\left(\frac{a^2}{8n^3\pi^3}\right)\right\} - \left\{0 - 0 + 2\left(\frac{a^3}{8n^3\pi^3}\right)\right\}\right]$$

$$= \frac{2}{a}\left[-\frac{a^3}{2n\pi}\right] = -\frac{a^2}{n\pi}$$

$$\therefore \quad x^2 = \frac{a^2}{3} + \frac{a^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{2\pi x}{a} + \frac{1}{2^2}\cos\frac{4\pi x}{a} + \ldots\right] - \frac{a^2}{\pi}\left[\frac{1}{1}\sin\frac{2\pi x}{a} + \frac{1}{2}\sin\frac{4\pi x}{a} + \ldots\right]$$

For deduction put $x = 0$ and $x = a$.

$$\therefore \quad 0 = \frac{a^2}{3} + \frac{a^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

i.e. $-\frac{1}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$ .... (i)

and $a^2 = \frac{a^2}{3} + \frac{a^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$

i.e. $\frac{2}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$ .... (ii)

Adding (i) and (ii), we get,

$$\frac{1}{3}=\frac{2}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\dots\right]$$

$$\therefore \quad \frac{\pi^2}{6}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\dots$$

Ex. 2 : Find the Fourier expansion of $f(x) = 2x - x^2$, $0 \le x \le 3$ whose period is 3. Also plot the graph of the function. **(M.U. 2005)**

Sol. : Here period $2l = 3 \quad \therefore \; l = 3/2$.

$$\text{Let} \quad f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum a_n \cos\frac{2n\pi x}{3} + \sum b_n \sin\frac{2n\pi x}{3} \qquad \dots\dots\dots (1)$$

$$\text{Now} \quad a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{3}\int_0^3 (2x-x^2)\,dx$$

$$= \frac{1}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = \frac{1}{3}\left[9-\frac{27}{3}\right] = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{(3/2)}\int_0^3 (2x-x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x-x^2)\cdot\left(\frac{3}{2n\pi}\cdot\sin\frac{2n\pi x}{3}\right)\right.$$

$$\left. -(2-2x)\left(-\frac{9}{4n^2\pi^2}\cdot\cos\frac{2n\pi x}{3}\right)+(-2)\left(-\frac{27}{8n^3\pi^3}\cdot\sin\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\left\{0-4\cdot\frac{9}{4n^2\pi^2}\cos 2n\pi + 0\right\} - \left\{0+2\cdot\frac{9}{4n^2\pi^2}+0\right\}\right]$$

$$= \frac{2}{3}\cdot\frac{9}{4n^2\pi^2}[-4-2] = -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{(3/2)}\int_0^3 (2x-x^2)\cdot\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x-x^2)\cdot\left(-\frac{3}{2n\pi}\cdot\cos\frac{2n\pi x}{3}\right)\right.$$

$$\left. -(2-2x)\left(-\frac{9}{4n^2\pi^2}\cdot\sin\frac{2n\pi x}{3}\right)+(-2)\left(\frac{27}{8n^3\pi^3}\cdot\cos\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\left\{\frac{9}{2n\pi}-0-\frac{27}{4n^3\pi^3}\right\} - \left\{0-0-\frac{27}{4n^3\pi^3}\right\}\right]$$

$$= \frac{2}{3}\left[\frac{9}{2n\pi}\right] = \frac{3}{n\pi}.$$

Putting these values in (1),

$$f(x) = 0 + \sum\left(-\frac{9}{n^2\pi^2}\right)\cos\frac{2n\pi x}{3} + \sum\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

$$= -\frac{9}{\pi^2}\sum\frac{1}{n^2}\cos\frac{2n\pi x}{3} + \frac{3}{\pi}\sum\frac{1}{n}\sin\frac{2n\pi x}{3}$$

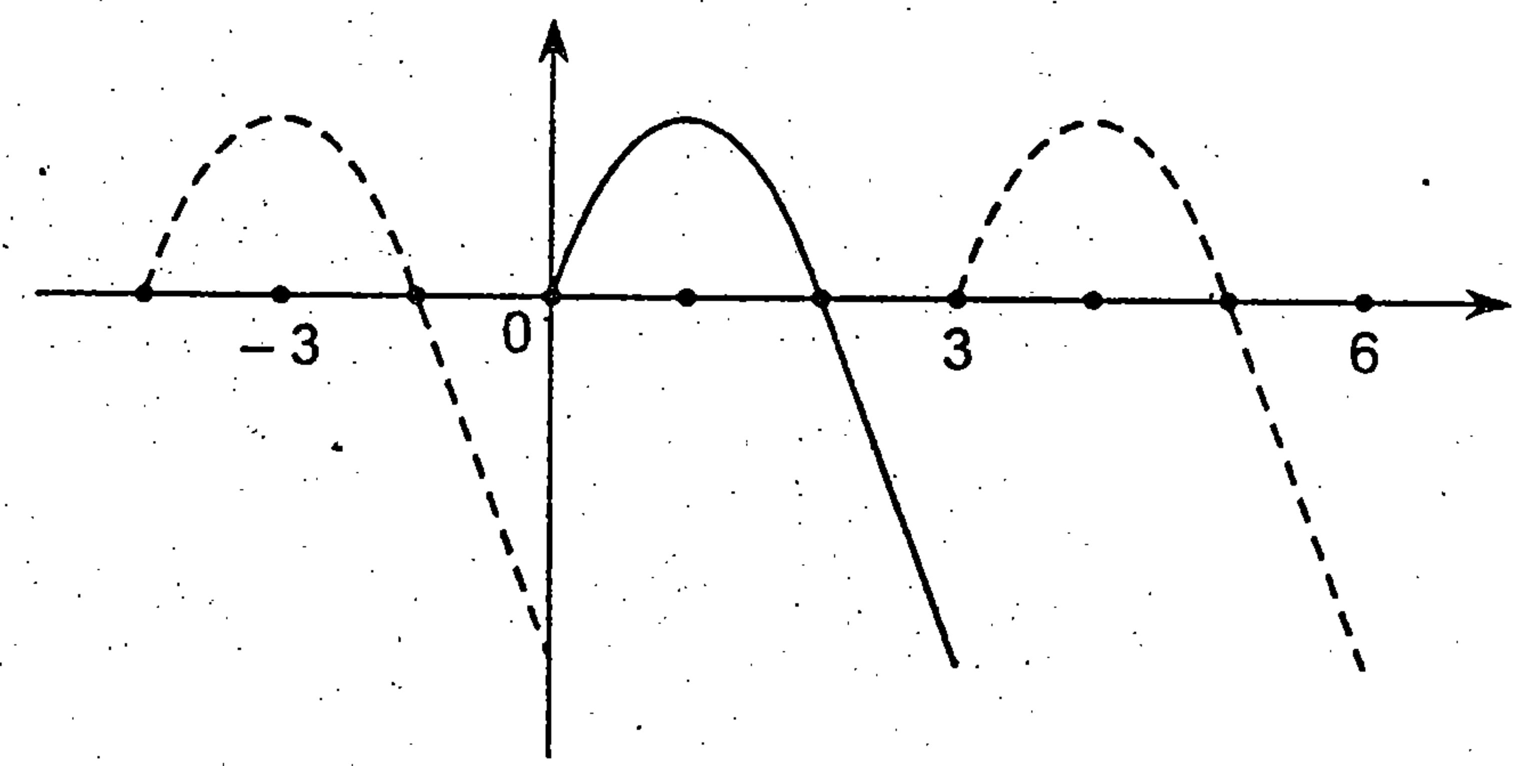
For the graph we see that $y = 2x - x^2$ is a parabola.

Now $y - 1 = -1 + 2x - x^2 = -(x-1)^2$

$\therefore \; Y = -X^2$, where $Y = y - 1$ and $X = x - 1$

When $x = 0$, $y = 0$; when $x = 2$, $y = 0$, When $x = 3$, $y = -3$.

Thus, we get, the following graph.



Ex. 3 : Find the Fourier expansion of $f(x) = 4 - x^2$ in the interval (0, 2). Graph the function and also state the values of the series for $x = 0, 1, 2, 10, 11$. **(M.U. 2002, 04)**

Hence, deduce that $\dfrac{\pi^2}{6}=\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dots\dots$ **(M.U. 2004)**

Sol. : Here $2l = 2 \quad \therefore \; l = 1$.

$$\text{Let} \quad f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$$

$$\text{i.e.} \quad f(x) = a_0 + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x \qquad \dots\dots\dots (1)$$

$$\therefore \quad a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\int_0^2 (4-x^2)\,dx$$

$$= \frac{1}{2}\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{1}{2}\left[8-\frac{8}{3}\right] = \frac{8}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \int_0^2 (4-x^2)\cos n\pi x\,dx$$

$$= \left[(4-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3}\right)\right]_0^2$$

$$= \left[\left\{0 - \frac{4}{n^2\pi^2} + 0\right\} - \left\{0 - 0 - 0\right\}\right] = -\frac{4}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^2 (4-x^2)\sin n\pi x\,dx$$

$$= \left[(4-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

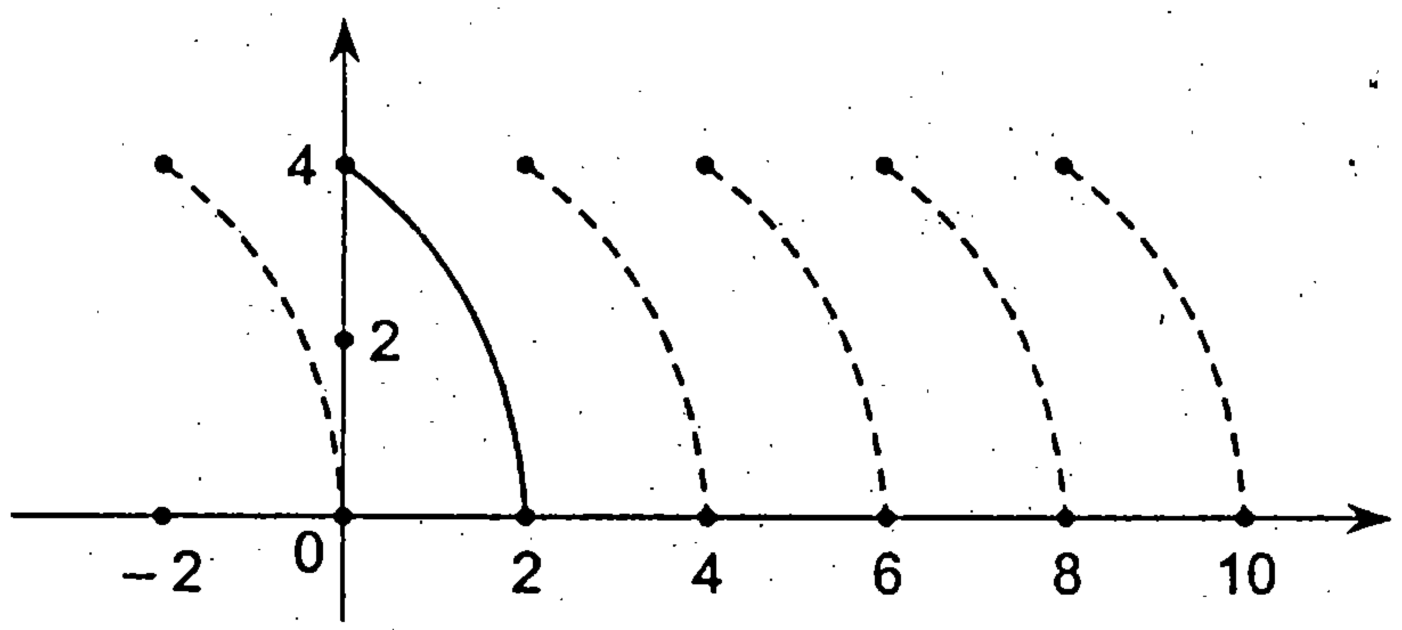
$$= \left[\left\{0 - 0 - \frac{2}{n^3\pi^3}\right\} - \left\{-\frac{4}{n\pi} - 0 - \frac{2}{n^3\pi^3}\right\}\right] = \frac{4}{n\pi}$$

$$\therefore\quad f(x) = 4 - x^2 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \ldots\right]$$

$$-\frac{4}{\pi}\left[\frac{1}{1}\sin\pi x + \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \ldots\right]$$

Now $y = 4 - x^2$ *i.e.* $y - 4 = -x^2$

i.e. $Y = -X^2$, where $Y = y - 4$ and $X = x$ is a parabola with vertex at (0, 4) and opening downwards as shown below.



Since, $f(x)$ is discontinuous at $x = 0, 2, 4, 6$ .... we find its value as follows.

$$f(c) = \frac{\lim\limits_{x\to c^-} f(x) + \lim\limits_{x\to c^+} f(x)}{2}$$

Now, for $c = 0$, from the graph

$$f(0) = \frac{\lim\limits_{x\to 0^-} f(x) + \lim\limits_{x\to 0^+} f(x)}{2} = \frac{0+4}{2} = 2$$

$$f(2) = \frac{\lim\limits_{x\to 2^-} f(x) + \lim\limits_{x\to 2^+} f(x)}{2} = \frac{4+0}{2} = 2$$

Similarly, $f(4) = f(6) = f(8) = f(10)$ .... $= 2$

Since, $f(x)$ is periodic with period 2.

Now, at $x = 1$, the function is continuous $\therefore\ f(1) = 4 - (1) = 3$

Also at $x = 11$, the function is continuous $\therefore\ f(11) = 3$.

Thus, we have $f(1) = 3,\ f(2) = 2,\ f(10) = 2,\ f(11) = 3$.

For deduction put $x = 0$ and $x = 2$.

$$\therefore\quad 4 - 0 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

$$\text{i.e.}\quad \frac{1}{3} = -\frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \quad \text{.......... (i)}$$

$$\text{and}\quad 4 - 4 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

$$\text{i.e.}\quad -\frac{2}{3} = -\frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \quad \text{.......... (ii)}$$

Adding (i) and (ii), we get,

$$-\frac{1}{3} = -\frac{2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

$$\therefore\quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Ex. 4 : Expand $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ period 2 into a Fourier Series.

(M.U. 1998, 2002)

Sol. : Here $2l = 2 \quad \therefore\ l = 1$

Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 0\,dx\right] = \frac{1}{2}\left[\frac{\pi x^2}{2}\right]_0^1 = \frac{\pi}{4}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\left[\int_0^1 \pi x \cos\frac{n\pi x}{1}\,dx + \int_1^2 0\cdot\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \left[\pi x\left(\frac{1}{n\pi}\sin n\pi x\right) - (\pi)\left(-\frac{1}{n^2\pi^2}\cos n\pi x\right)\right]_0^1$$

$$= \pi\cdot\frac{1}{n^2\pi^2}\left[(-1)^n - 1\right] = -\frac{1}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\left[\int_0^1 \pi x\sin\frac{n\pi x}{1}\,dx + \int_1^2 0\cdot\sin\frac{n\pi x}{1}\,dx\right]$$

$$= \left[\pi x\left(-\frac{1}{n\pi}\cos n\pi x\right) - (\pi)\left(-\frac{1}{n^2\pi^2}\sin n\pi x\right)\right]_0^1 = -\frac{1}{n}(-1)^n$$

$$\therefore\quad f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \ldots\right] - \left[-\frac{\sin\pi x}{1} + \frac{\sin 2\pi x}{2} - \frac{\sin 3\pi x}{3} + \ldots\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \ldots\right] + \left[\frac{1}{1}\cdot\sin\pi x - \frac{1}{2}\cdot\sin 2\pi x + \frac{1}{3}\cdot\sin 3\pi x - \ldots\right]$$

Ex. 5 : If $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$ with period 2, show that

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x \qquad \textbf{(M.U. 2003, 04)}$$

Sol. : Here $2l = 2 \quad \therefore\ l = 1$

Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

i.e. $f(x) = a_0 + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x$ .... (1)

$$\therefore\quad a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right]$$

$$= \frac{1}{2}\left[\pi\left\{\frac{x^2}{2}\right\}_0^1 + \pi\left\{2x - \frac{x^2}{2}\right\}_1^2\right]$$

$$= \frac{\pi}{2}\left[\left\{\frac{1}{2}\right\} + \left\{4 - 2 - 2 + \frac{1}{2}\right\}\right] = \frac{\pi}{2}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \int_0^1 \pi x\cdot\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \pi\left[\left\{x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right\}_0^1 + \left\{(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right\}_1^2\right]$$

$$= \pi\left[\left\{\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right\} + \left\{-\frac{1}{n^2\pi^2} + \frac{\cos n\pi}{n^2\pi^2}\right\}\right]$$

$$= \frac{2\pi}{n^2\pi^2}[\cos n\pi - 1] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \int_0^1 \pi x\cdot\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\left[\left\{x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right\}_0^1 + \left\{(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right\}_1^2\right]$$

$$= \pi\left[\left\{-\frac{\cos n\pi}{n\pi}\right\} + \left\{+\frac{\cos n\pi}{n\pi}\right\}\right] = 0$$

Putting these values in (1)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x$$

EXERCISE

Obtain the Fourier expansions of the following functions

1. $f(x) = 2 - \dfrac{x^2}{2}$ in $0 \le x \le 2$

$$\left[\textbf{Ans. : } f(x) = \frac{4}{3} - \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n^2}\cos n\pi x + \frac{2}{\pi}\cdot\sum\frac{1}{n}\cdot\sin n\pi x\right]$$

2. $f(x) = \dfrac{a}{2} - x,\ 0 < x < a$

$$\left[\textbf{Ans. : } f(x) = \frac{a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cdot\sin\frac{2n\pi x}{a}\right]$$

3. $f(x) = l - x,\ 0 < x < l$
$\quad\ = 0,\quad l < x < 2l.$

Hence, deduce that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$

$$\left[\textbf{Ans. : } f(x) = l\left[\frac{1}{4} + \frac{2}{\pi^2}\left(1 + \cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\ldots\right)\right.\right.$$
$$\left.\left. + \frac{1}{\pi}\left(\frac{1}{l}\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\ldots\right)\right]\right]$$

Then put $x = l/2$.]

4. $f(x) = 1,\quad 0 < x < 1$
$\quad\ = x,\quad 1 < x < 2$

$$\left[\textbf{Ans. : } f(x) = \frac{5}{4} - \frac{4}{\pi^2}\left[\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2}\right.\right.$$
$$\left.\left. + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \frac{1}{4^2}\cos\frac{4\pi x}{2} + \ldots\ldots\right]\right]$$

5. $f(x) = c,\quad 0 < x < a$
$\quad\ = 0,\quad a < x < l$

$$\left[\textbf{Ans. : } f(x) = \frac{ca}{l} + \frac{2c}{\pi}\left[\sin\left(\frac{\pi a}{l}\right)\cos\left(\frac{\pi x}{l}\right)\right.\right.$$
$$\left.\left. + \frac{1}{2}\sin\left(\frac{2\pi a}{l}\right)\cos\left(\frac{2\pi x}{l}\right) + \ldots\right]\right]$$

6. (a) $f(x) = \begin{cases} kx, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ **(M.U. 1998)**

(b) $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ **(M.U. 2002)**

$$\left[\textbf{Ans. : (a) } f(x) = \frac{k}{4} - \frac{2k}{\pi^2}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right)\right.$$
$$+ \frac{k}{\pi}\left(\frac{\sin\pi x}{1} + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \ldots\right)$$

(b) In (a) put $k = \pi$.

$$\left. f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \ldots\right) + \left(\frac{\sin\pi x}{1} + \frac{\sin 2\pi x}{2} + \ldots\right)\right]$$

7. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1 - x, & 1 < x < 2 \end{cases}$

$$\left[\textbf{Ans. : } f(x) = -\frac{4}{\pi^2}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)\right.$$
$$\left. + \frac{2}{\pi}\left(\frac{\sin\pi x}{1} + \frac{\sin 3\pi x}{3} + \ldots\right)\right]$$

8. $f(x) = \begin{cases} x, & 0 < x < c/2 \\ c - x, & c/2 < x < c \end{cases}$

$$\left[\textbf{Ans. : } f(x) = \frac{4c}{\pi^2}\left[\frac{1}{1^2}\sin\left(\frac{\pi x}{c}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{c}\right)\right.\right.$$
$$\left.\left. + \frac{1}{5^2}\sin\left(\frac{5\pi x}{c}\right) - \ldots\right]\right]$$

9. $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & x = 1 \\ \pi(x - 2), & 1 < x < 2 \end{cases}$

Hence, show that $\dfrac{\pi}{4} = \dfrac{1}{1} - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7}$. **(M.U. 1997, 2002, 03)**

$$\left[\textbf{Ans. : } f(x) = \frac{\pi}{4} + 2\cdot\sum_{n=1}^{\infty}(-1)^{n+1}\sin n\pi x. \quad \text{Then put } x = \frac{1}{2}\right]$$

10. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & x > 2 \end{cases}$

(**Hint :** In example 8 above put $c = 2$)

$$\left[\textbf{Ans. : } f(x) = \frac{8}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - \ldots\right)\right]$$

11. $f(x) = \begin{cases} -x, & 0 < x < l \\ 0, & l \le x < 2l \end{cases}$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

[**Ans. :** $f(x) = -\dfrac{l}{4} + \dfrac{2l}{\pi^2}\left[\dfrac{1}{1}\cos\dfrac{\pi x}{l} + \dfrac{1}{3}\cos\dfrac{3\pi x}{l} + \ldots\right]$

$-\dfrac{l}{\pi}\left[\dfrac{1}{1}\sin\dfrac{\pi x}{l} - \dfrac{1}{2}\sin\dfrac{2\pi x}{l} + \dfrac{1}{3}\sin\dfrac{3\pi x}{l}\right]$

Then put $x = 0$.]

12. $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$

Hence, deduce that $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots = \dfrac{\pi^2}{8}$. **(M.U. 2004)**

$$\left[\textbf{Ans. : } f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum\left\{\frac{1}{(2n-1)^2}\cos(2n-1)\pi x\right\}\right]$$

13. $f(x) = \begin{cases} 3kx/l, & 0 < x < (l/3) \\ 3k(l-2x)/l, & (l/3) < x < (2l/3) \\ 3k(x-l)/l, & (2l/3) < x < l \end{cases}$ **(M.U. 2003)**

$$\left[\textbf{Ans. : } \frac{9k}{\pi^2}\sum\frac{1}{n^2}\sin\frac{2n\pi}{3}\cdot\sin\frac{2n\pi x}{l}\right]$$

(b) Fourier expansion in the interval $(-l, l)$

Ex. 1 : Find the Fourier expansion of $f(x) = e^{ax}$ in $(-l, l)$.

Sol. : Let $f(x) = a_0 + \sum a_n \cos\dfrac{n\pi x}{l} + \sum b_n \sin\dfrac{n\pi x}{l}$ .... (1)

$$\therefore \quad a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2l}\int_{-l}^{l} e^{ax}\,dx$$

$$= \frac{1}{2l}\left[\frac{e^{ax}}{a}\right]_{-l}^{l} = \frac{1}{2al}\left[e^{al} - e^{-al}\right]$$

$$= \frac{1}{al}\left[\frac{e^{al} - e^{-al}}{2}\right] = \frac{\sin h\,al}{al}$$

$$\therefore \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{ax}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{1}{a^2 + n^2\pi^2/l^2}\left\{e^{ax}\left(a\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right\}\right]_{-l}^{l}$$

$$= \frac{l}{a^2l^2 + n^2\pi^2}\left[e^{al}(a\cos n\pi + 0) - e^{-al}(a\cos n\pi + 0)\right]$$

$$= \frac{a\,l}{a^2l^2 + n^2\pi^2}\left[e^{al}(-1)^n - e^{-al}(-1)^n\right]$$

$$= \frac{2\,a\,l\,(-1)^n}{a^2l^2 + n^2\pi^2}\left[\frac{e^{al} - e^{-al}}{2}\right] = \frac{2\,al\,(-1)^n}{a^2l^2 + n^2\pi^2}\sin h\,a\,l.$$

$$\therefore \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l} e^{ax}\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{1}{a^2 + n^2\pi^2/l^2}\left\{e^{ax}\left(a\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right\}\right]_{-l}^{l}$$

$$= \frac{l}{a^2l^2 + n^2\pi^2}\left[e^{al}\left(0 - \frac{n\pi}{l}\cos n\pi\right) - e^{-al}\left(0 - \frac{n\pi}{l}\cos n\pi\right)\right]$$

$$= \frac{l}{a^2l^2 + n^2\pi^2}\left[e^{al}\left(0 - \frac{n\pi}{l}(-1)^n\right) - e^{-al}\left(0 - \frac{n\pi}{l}(-1)^n\right)\right]$$

$$= \frac{l}{a^2l^2 + n^2\pi^2}\cdot\frac{n\pi}{l}\cdot(-1)^{n+1}(e^{al} - e^{-al})$$

$$= \frac{2\,n\,\pi\,(-1)^{n+1}}{a^2l^2 + n^2\pi^2}\cdot\left(\frac{e^{al} - e^{-al}}{2}\right) = \frac{2\,n\,\pi\,(-1)^{n+1}}{a^2l^2 + n^2\pi^2}\cdot\sin h\,a\,l.$$

Putting these values in (1)

$$f(x) = \frac{\sin h\,a\,l}{a\,l} + \sum\frac{2\,al\,(-1)^n}{a^2l^2 + n^2\pi^2}\sin h\,a\,l\cos\frac{n\pi x}{l}$$

$$+ \sum\frac{2\,n\,\pi\,(-1)^{n+1}}{a^2l^2 + n^2\pi^2}\cdot\sin h\,a\,l\sin\frac{n\pi x}{l}$$

$$= \frac{\sin h\,a\,l}{a\,l} + 2al\sin h\,a\,l\sum\frac{(-1)^n}{a^2l^2 + n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$+ 2\pi\sin h\,a\,l\sum\frac{(-1)^{n+1}}{a^2l^2 + n^2\pi^2}\sin\frac{n\pi x}{l}.$$

Ex. 2 : Find the Fourier expansion of $f(x) = \begin{cases} 2, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$

Sol. : Comparing the interval $(-2, 2)$ with $(-l, l)$ we find that here $l = 2$.

Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

$$= a_0 + \sum a_n \cos\frac{n\pi x}{2} + \sum b_n \sin\frac{n\pi x}{2} \qquad \text{.......... (1)}$$

$$\therefore \quad a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{4}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{4}\left[\int_{-2}^{0} 2\,dx + \int_0^2 x\,dx\right] = \frac{1}{4}\left[\left\{2x\right\}_{-2}^{0} + \left\{\frac{x^2}{2}\right\}_0^2\right]$$

$$= \frac{1}{4}\left[\{0+4\} + \{2-0\}\right] = \frac{3}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \int_0^2 x\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left\{2\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right\}_{-2}^{0} + \left\{(x)\cdot\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}_0^2\right]$$

$$= \frac{1}{2}\left[\left\{0-0\right\} + \left\{0 + \frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\right\}\right]$$

$$= \frac{2}{n^2\pi^2}\left\{\cos n\pi - 1\right\} = \begin{cases} 0 & \text{when } n \text{ is even} \\ -4 & \text{when } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \int_0^2 x\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left\{2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\right\}_{-2}^{0} + \left\{(x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}_0^2\right]$$

$$= \frac{1}{2}\left[\left\{-\frac{4}{n\pi} + \frac{4}{n\pi}\cos n\pi\right\} + \left\{(2)\left(-\frac{2}{n\pi}\cos n\pi\right) + 0\right\}\right]$$

$$= \frac{1}{2}\left\{-\frac{4}{n\pi} + \frac{4}{n\pi}\cos n\pi - \frac{4}{n\pi}\cos n\pi\right\} = -\frac{2}{n\pi}$$

Putting these values in (1)

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots..\right]$$
$$- \frac{2}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{2} + \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \ldots..\right]$$

Ex. 3 : Expand $f(x) = \begin{cases} 0, & -c < x < 0 \\ a, & 0 < x < c \end{cases}$ in a Fourier series of period $2a$.

(M.U. 2002)

Sol. : Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$ and $l = c$ .... (1)

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx$$

$$= \frac{1}{2c}\left[\int_{-c}^{0} 0\cdot dx + \int_a^c a\,dx\right] = \frac{1}{2c}\left[0 + \left[\,ax\,\right]_0^c\right] = \frac{1}{2c}\cdot ac = \frac{a}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{c}\left[\int_{-c}^{0} 0\cdot\cos\frac{n\pi x}{c}\,dx + \int_0^c a\cos\frac{n\pi x}{c}\,dx\right]$$

$$= \frac{1}{c}\left[a\left(\frac{c}{n\pi}\sin\frac{n\pi x}{c}\right)\right]_0^c = \frac{a}{n\pi}[0-0] = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{\pi x}{l}\,dx$$

$$= \frac{1}{c}\left[\int_{-c}^{0} 0\cdot\sin\frac{n\pi x}{c}\,dx + \int_0^c a\sin\frac{n\pi x}{c}\,dx\right]$$

$$= \frac{1}{c}\left[a\left(-\frac{c}{n\pi}\right)\cos\frac{n\pi x}{c}\right]_0^c$$

$$= -\frac{a}{n\pi}\left\{\cos n\pi - 1\right\} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2a}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

Putting these values in (1)

$$f(x) = \frac{a}{2} + \frac{2a}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \ldots..\right]$$

Ex. 4 : A sinusoidal voltage $E \sin \omega x$ is passed through a half-wave rectifier which clips of the wave and the resulting function is given by

$$f(x) = \begin{cases} 0 \quad , & -\frac{\pi}{\omega} < x < 0 \\ E \sin \omega x, & 0 < x < \frac{\pi}{\omega} \end{cases}$$

with period $\frac{2\pi}{\omega}$. Find the Fourier expansion of $f(x)$.

Sol. : Here $2l = \frac{2\pi}{\omega} \quad \therefore \; l = \frac{\pi}{\omega}$

Let $f(x) = a_0 + \sum a_n \cos \frac{n\pi x}{l} + \sum b_n \sin \frac{n\pi x}{l}$

$$= a_0 + \sum a_n \cos n\omega x + \sum b_n \sin n\omega x \quad \text{.......... (1)}$$

$$a_0 = \frac{1}{2l} \int_{-l}^{l} f(x) dx = \frac{1}{(2\pi/\omega)} \left[\int_{-\pi/\omega}^{0} 0 \cdot dx + \int_{0}^{\pi/\omega} E \sin \omega x \, dx \right]$$

$$= \frac{\omega}{2\pi} \left[-\frac{E \cos \omega x}{\omega} \right]_0^{\pi/\omega} = \frac{-E}{2\pi}[-1-1] = \frac{E}{\pi}$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(x) \cos n\omega x \, dx$$

$$= \frac{\omega}{\pi} \left[\int_{-\pi/\omega}^{0} 0 \cdot dx + \int_{0}^{\pi/\omega} E \sin \omega x \cdot \cos n\omega x \, dx \right]$$

$$= \frac{E\omega}{2\pi} \int_0^{\pi/\omega} \{\sin (1+n) \omega x + \sin (1-n) \omega x\} dx$$

$$= \frac{E\omega}{2\pi} \left[\left\{ -\frac{\cos (1+n) \omega x}{(1+n) \omega} - \frac{\cos (1-n) \omega x}{(1-n) \omega} \right\}_0^{\pi/\omega} \right]$$

$$= \frac{E\omega}{2\pi} \left[-\frac{\cos (1+n) \pi}{1+n} - \frac{\cos (1-n) \pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n} \right]$$

$$= \frac{E}{2\pi} \left[\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n} \right]$$

$$[\because \cos (1 \pm n) \pi = -\cos n\pi]$$

$$= \frac{E}{2\pi} (1 + \cos n\pi) \left[\frac{1}{1+n} + \frac{1}{1-n} \right] = \frac{E}{2\pi} \left[1 + (-1)^n \right] \cdot \frac{2}{1-n^2}$$

$$= \begin{cases} \frac{2E}{\pi} \cdot \frac{1}{1-n^2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd and } n \neq 1 \end{cases}$$

When $n = 1$, we have

$$a_1 = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{\pi x}{c} dx = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(x) \cos \omega x \, dx$$

$$= \frac{\omega}{\pi} \left[\int_{-\pi/\omega}^{0} 0 \cdot dx + \int_{0}^{\pi/\omega} E \sin \omega x \cdot \cos \omega x \, dx \right]$$

$$= \frac{\omega E}{2\pi} \int_0^{\pi/\omega} \sin 2\omega x \, dx = \frac{\omega E}{2\pi} \left[-\frac{\cos 2\omega x}{2\pi} \right]_0^{\pi/\omega} = -\frac{\omega E}{2\pi} [1-1] = 0$$

Now, $\quad b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx = \frac{\omega}{\pi} \int_{-l}^{l} f(x) \sin n\omega x \, dx$

$$= \frac{\omega}{\pi} \left[\int_{-\pi/\omega}^{0} 0 \cdot dx + \int_{0}^{\pi/\omega} E \sin \omega x \sin n\omega x \, dx \right]$$

$$= -\frac{E\omega}{2\pi} \int_0^{\pi/\omega} \left[\cos (1+n) \omega x - \cos (1-n) \omega x \right] dx$$

$$= -\frac{E\omega}{2\pi} \left[\frac{\sin (1+n) \omega x}{(1+n) \omega} - \frac{\sin (1-n) \omega x}{(1-n) \omega} \right]_0^{\pi/\omega}$$

$$= -\frac{E}{2\pi} \left[\frac{\sin (1+n) \pi}{1+n} - \frac{\sin (1-n) \pi}{1-n} - 0 \right]$$

$$= 0 \text{ except when } n = 1 \qquad [\because \sin (1 \pm n) \pi = 0]$$

When $n = 1$, we have

$$b_1 = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{\pi x}{l} dx = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(x) \sin \omega x \, dx$$

$$= \frac{\omega}{\pi} \left[\int_{-\pi/\omega}^{0} 0 \cdot dx + \int_{0}^{\pi/\omega} E \sin \omega x \cdot \sin \omega x \, dx \right]$$

$$= \frac{\omega E}{\pi} \int_0^{\pi/\omega} \sin^2 \omega x \, dx = \frac{\omega E}{\pi} \int_0^{\pi/\omega} \left(\frac{1 - \cos 2\omega x}{2} \right) dx$$

$$= \frac{\omega E}{2\pi} \left[x - \frac{\sin 2\omega x}{2} \right]_0^{\pi/\omega} = \frac{\omega E}{2\pi} \left[\frac{\pi}{\omega} \right] = \frac{E}{2}$$

Hence, putting these values in (1),

$$f(x) = \frac{E}{\pi} - \frac{2E}{\pi} \left[\frac{1}{(2^2-1)} \cos 2\omega x + \frac{1}{(4^2-1)} \cos 4\omega x + \frac{1}{(6^2-1)} \cos 6\omega x + \ldots \right] + \frac{E}{2} \sin \omega x$$

$$= \frac{E}{\pi} - \frac{2E}{\pi} \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)} \cos 2n \omega x + \frac{E}{2} \sin \omega x$$

EXERCISE

Find the Fourier Series of the following functions.

1. $f(x) = \begin{cases} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases}$ period 10. **(M.U. 2003)**

$$\left[\textbf{Ans. : } f(x) = \frac{3}{2} + \frac{6}{\pi}\left[\frac{1}{1}\cdot\sin\frac{\pi x}{5} + \frac{1}{3}\cdot\sin\frac{3\pi x}{5} + \frac{1}{5}\cdot\sin\frac{5\pi x}{5} + \ldots\right]\right]$$

(**Hint :** See solved ex. 3 above. Here $a = 3$, $c = 5$.)

2. $f(x) = \begin{cases} 2x, & 0 \le x < 3 \\ 0, & -3 < x < 0 \end{cases}$ period 6.

$$\left[\textbf{Ans. : } f(x) = \frac{3}{2} - \frac{12}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{3} + \frac{1}{3^2}\cos\frac{3\pi x}{3} + \frac{1}{5^2}\cos\frac{5\pi x}{3} + \ldots\right] + \frac{1}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{3} - \frac{1}{2}\sin\frac{2\pi x}{3} + \frac{1}{3}\sin\frac{3\pi x}{3} - \ldots\right]\right]$$

3. $f(x) = \begin{cases} x, & -1 < x < 0 \\ x+2, & 0 < x < 1 \end{cases}$ period 2. **(M.U. 2004)**

$$\left[\textbf{Ans. : } f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[1 - 2(-1)^n\right]\sin n\pi x\right]$$

4. $f(x) = \begin{cases} l, & -l < x < 0 \\ x, & 0 < x < l \end{cases}$

$$\left[\textbf{Ans. : } f(x) = \frac{3l}{4} - \frac{2l}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{l}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{l}\right) + \ldots\right] - \frac{1}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \ldots\right]\right]$$

5. $f(x) = \begin{cases} -x, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$ **(M.U. 2004)**

$$\left[\textbf{Ans. : } 1 - \frac{4}{\pi^2}\sum\frac{1}{(2n-1)^2}\cos n\pi x\right]$$

6. $f(x) = |x|, \ -2 < x < 2$

Hence, deduct that $\sum\dfrac{1}{(2n-1)^4} = \dfrac{\pi^4}{96}$. **(M.U. 2001, 03)**

$$\left[\textbf{Ans. : } 1 - \frac{8}{\pi^2}\sum\frac{1}{(2n-1)^2}\cos\left[\frac{(2n-1)\pi x}{2}\right]\right]$$

(c) Even and Odd functions in the interval $(-l, l)$

(i) Even function in $(-l, l)$: If $f(x)$ is even in the interval $(-l, l)$ then,

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{2}{2l}\int_0^l f(x)\,dx = \frac{1}{l}\int_0^l f(x)\,dx \qquad [\because f(x) \text{ is even}]$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$\left[\because f(x)\cos\frac{n\pi x}{l} \text{ is even}\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 0 \qquad \left[\because f(x)\sin\frac{n\pi x}{l} \text{ is odd}\right]$$

(ii) Odd function in $(-l, l)$: If $f(x)$ is odd in the interval $(-l, l)$ then,

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = 0 \qquad [\because f(x) \text{ is odd}]$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = 0 \qquad \left[\because f(x)\cos\frac{n\pi x}{l} \text{ is odd}\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$\left[\because f(x)\sin\frac{n\pi x}{l} \text{ is even}\right]$$

Thus, we have

If $f(x)$ is even in $(-l, l)$

$$\boxed{a_0 = \frac{1}{l}\int_0^l f(x)\,dx,\ \ a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx,\ \ b_n = 0}$$

If $f(x)$ is odd in $(-l, l)$

$$\boxed{a_0 = 0,\ \ a_n = 0,\ \ b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx}$$

Ex. 1 : Obtain the Fourier expansion of x^2 from $x = -l$ to $x = l$ and hence, deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^2} + \ldots \qquad \textbf{(M.U. 1995)}$$

Sol. : Since, $f(x) = x^2$ is an even function, we have $b_n = 0$.

Now, $a_0 = \dfrac{1}{l}\displaystyle\int_0^l f(x)\,dx = \frac{1}{l}\int_0^l x^2\,dx = \frac{1}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{l^2}{3}$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x^2\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{2l^3}{n^2\pi^2}(-1)^n\right] = (-1)^n\frac{4l^2}{n^2\pi^2}$$

$$\therefore \quad x^2 = \frac{l^2}{3} + \sum(-1)^n\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right]$$

Now, put $x = 0$.

$$\therefore \quad 0 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

$$\therefore \quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

Ex. 2 : Find the Fourier Series for $f(x) = 1 - x^2$ in $(-1, 1)$. **(M.U. 2004)**

Sol. : We have $f(-x) = 1 - (-x^2) = 1 - x^2 = f(x)$. Hence, $f(x)$ is even.

Comparing $(-l, l)$ with $(-1, 1)$ we get $l = 1$

Now, $a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \int_0^1 (1 - x^2)\,dx$

$$= \left[x - \frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = 2\int_0^1 (1 - x^2)\cos n\pi x\,dx$$

$$= 2\left[(1 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[-\frac{2\cos n\pi}{n^2\pi^2}\right] = \frac{-4(-1)^n}{n^2\pi^2}$$

Hence, $f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$

Ex. 3 : Obtain the Fourier Series for $f(x)$, where $f(x) = \begin{cases} -c, & -a < x < 0 \\ c, & 0 < x < a \end{cases}$

Sol. : We have

$$f(-x) = \begin{cases} -c, & -a < -x < 0 \\ c, & 0 < -x < a \end{cases} = \begin{cases} -c, & a > x > 0 \\ c, & 0 > x > -a \end{cases}$$

$$= \begin{cases} c, & -a < x < 0 \\ -c, & 0 < x < a \end{cases} = -\begin{cases} -c, & -a < x < 0 \\ c, & 0 < x < a \end{cases} = -f(x)$$

Hence, $f(x)$ is odd function

$$\therefore \quad a_0 = 0, \quad a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{a}\int_0^a c \cdot \sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2c}{a}\left[-\frac{a}{n\pi}\cos\frac{n\pi x}{a}\right]_0^a = \frac{2c}{a}\left(-\frac{a}{n\pi}\right)\left[(-1)^n - 1\right]$$

$$= -\frac{2c}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & \text{when } n \text{ is even} \\ \dfrac{4c}{n\pi} & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore \quad f(x) = \frac{4c}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{a} + \frac{1}{3}\sin\frac{3\pi x}{a} + \frac{1}{5}\sin\frac{5\pi x}{a} + \ldots\right]$$

Ex. 4 : Obtain Fourier expansion of $\sin ax$ in the interval $-l < x < l$, where a is not an integer. **(M.U. 2002)**

Sol. : Here $f(-x) = \sin(-ax) = -\sin ax = -f(x)$. Hence, $f(x)$ is an odd function. [$\therefore a_n = 0$].

Let $\sin ax = \sum b_n \sin\frac{n\pi x}{l}$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l} = \frac{2}{l}\int_0^l \sin ax \sin\frac{n\pi x}{l}\,dx$$

$$= -\frac{1}{l}\int_0^l\left[\cos\left(a + \frac{n\pi}{l}\right)x - \cos\left(a - \frac{n\pi}{l}\right)x\right]dx$$

$$= -\frac{1}{l}\left[\frac{l}{al + n\pi}\sin\left(a + \frac{n\pi}{l}\right)x - \frac{l}{al - n\pi}\sin\left(a - \frac{n\pi}{l}\right)x\right]_0^l$$

$$= -\left[\frac{l}{al+n\pi}\sin(al+n\pi) - \frac{l}{al-n\pi}\sin(al-n\pi)\right]$$

$$= -\left[\frac{\sin al\cos n\pi}{al+n\pi} - \frac{\sin al\cos n\pi}{al-n\pi}\right]$$

$$= \frac{(-2n\pi)}{(a^2l^2-n^2\pi^2)}\sin al\cos n\pi = -\frac{2n\pi(-1)^n}{n^2\pi^2-a^2l^2}\sin al$$

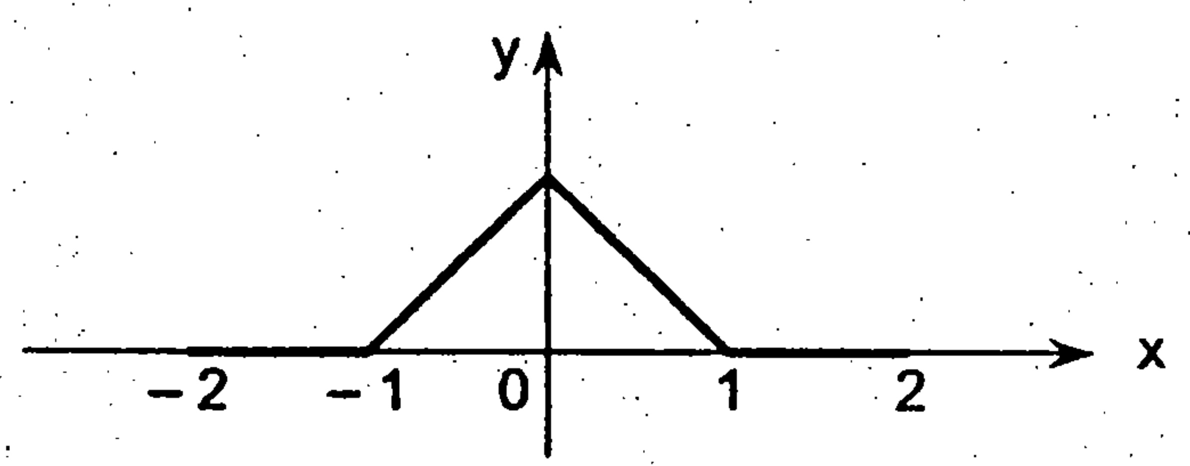
$$\therefore\quad f(x) = 2\pi\sin al\sum\frac{(-n)(-1)^n}{n^2\pi^2-a^2l^2}\sin\frac{n\pi x}{l}$$

$$= 2\pi\sin al\left[\frac{1}{(1^2\pi^2-a^2l^2)}\sin\frac{\pi x}{l} - \frac{2}{(2^2\pi^2-a^2l^2)}\sin\frac{2\pi x}{l} + \frac{3}{(3^2\pi^2-a^2l^2)}\sin\frac{3\pi x}{l} - \ldots\right]$$

Ex. 5 : Find the Fourier expansion of $f(x) = \begin{cases} 0, & -2<x<-1 \\ 1+x, & -1<x<0 \\ 1-x, & 0<x<1 \\ 0, & 1<x<2 \end{cases}$

(M.U. 1993, 2002, 03)

Sol. : The graph of the function is shown below,



$$\text{Further } f(-x) = \begin{cases} 0, & -2<-x<-1 \\ 1-x, & -1<-x<0 \\ 1+x, & 0<-x<1 \\ 0, & 1<-x<2 \end{cases}$$

$$= \begin{cases} 0, & 2>x>1 \\ 1-x, & 1>x>0 \\ 1+x, & 0>x>-1 \\ 0, & -1>x>-2 \end{cases} = \begin{cases} 0, & -2<x<-1 \\ 1+x, & -1<x<0 \\ 1-x, & 0<x<1 \\ 0, & 1<x<2 \end{cases}$$

$$= f(x)$$

$\therefore$ $f(x)$ is an even function.

$\therefore$ $b_n = 0$ and $l = 2$.

$$\therefore\quad f(x) = a_0 + \sum a_n\cos\frac{n\pi x}{l}$$

$$= a_0 + \sum a_n\cos\frac{n\pi x}{2}$$

$$\therefore\quad a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{2}\int_0^2 f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^1(1-x)\,dx + \int_1^2 0\cdot dx\right]$$

$$= \frac{1}{2}\left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1(1-x)\cos\frac{n\pi x}{2}\,dx + \int_1^2 0\cdot dx$$

$$= \left[(1-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^1$$

$$= \left[\left(0 - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right) - \left(0 - \frac{4}{n^2\pi^2}\right)\right]$$

$$= \frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)$$

$$\therefore\quad f(x) = \frac{1}{4} + \frac{4}{\pi^2}\sum\frac{1}{n^2}\left(1-\cos\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}$$

Ex. 6 : Find Fourier expansion for $f(x) = x - x^2,\ -1 < x < 1$. **(M.U. 2005)**

Sol. : The given function is the difference between odd and even functions which can be written as

$$f(x) = f_1(x) - f_2(x)$$

where, $f_1(x) = x$, is an odd function and

$f_2(x) = x^2$, is an even function.

Here $l = 1$.

Now, for $f_1(x) = x$ which is odd, $a_n = 0$ and

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{1}\int_0^1 x\sin n\pi x\,dx$$

$$= 2\left[x\left(-\frac{1}{n\pi}\cos n\pi x\right) - (1)\left(-\frac{1}{n^2\pi^2}\sin n\pi x\right)\right]_0^1$$

$$= 2\left[-\frac{1}{n\pi}(-1)^n\right] = -\frac{2}{n\pi}(-1)^n$$

Further for $f_2(x) = x^2$ which is even, $b_n = 0$.

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{1}\int_0^1 x^2 dx = \frac{1}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{c}\,dx = 2\int_0^1 x^2\cos n\pi x\,dx$$

$$= 2\left[x^2\left(\frac{1}{n\pi}\sin n\pi x\right) - 2x\left(-\frac{1}{n^2\pi^2}\cos n\pi x\right) + 2\left(-\frac{1}{n^3\pi^3}\sin n\pi x\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}(-1)^n$$

$$\therefore \quad f(x) = f_1(x) - f_2(x) = -f_2(x) + f_1(x)$$

$$\therefore \quad f(x) = -\frac{1}{3} - \frac{4}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos n\pi x - \frac{2}{\pi}\sum\frac{(-1)^n}{n}\sin n\pi x$$

EXERCISE

Obtain Fourier Series for the following functions.

1. $f(x) = a^2 - x^2$ in $(-a, a)$ **(M.U. 2004)**

$$\left[\textbf{Ans. :}\ f(x) = \frac{2a^2}{3} + \frac{4a^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{a} - \frac{1}{2^2}\cos\frac{2\pi x}{a} + \frac{1}{3^2}\cos\frac{3\pi x}{a} - \ldots\right]\right]$$

2. $f(x) = \sin 2x$ in $(-l, l)$

(**Hint :** Put $a = 2$ in the solved ex. 4 above.)

$$\left[\textbf{Ans. :}\ f(x) = 2\pi\sin 2l\left[\frac{1}{\pi^2 - 2^2 l^2}\sin\frac{\pi x}{l} - \frac{2}{2^2\pi^2 - 2^2 l^2}\sin\frac{2\pi x}{l} + \ldots\right]\right]$$

3. $f(x) = \begin{cases} -\sin\dfrac{\pi x}{c}, & -c < x < 0 \\ \sin\dfrac{\pi x}{c}, & 0 < x < c \end{cases}$ **(M.U. 1999)**

$$\left[\textbf{Ans. :}\ f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{2^2 - 1} + \frac{\cos 4x}{4^2 - 1} + \ldots\right]\right]$$

4. $f(x) = x^2, \quad -1 < x < 1$

(**Hint :** Put $l = 1$ in the solved ex. 1 above.)

$$\left[\textbf{Ans. :}\ f(x) = \frac{1}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x - \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \ldots\right]\right]$$

5. $f(x) = 9 - x^2$ in $(-3, 3)$. **(M.U. 2002)**

(**Hint :** Put $a = 3$ in the above example 1 of this exercise.)

$$\left[\text{Ans. :}\ f(x) = 6 + \frac{36}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{3} - \frac{1}{2^2}\cos\frac{2\pi x}{3} + \frac{1}{3^2}\cos\frac{3\pi x}{3} - \ldots\right]\right]$$

6. $f(x) = x + x^2$ in $(-1, 1)$ **(M.U. 1994)**

$$\left[\textbf{Ans. :}\ f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos n\pi x - \frac{2}{\pi}\sum\frac{(-1)^n}{n}\sin n\pi x\right]$$

7. $f(x) = x - x^3$ in $(-1, 1)$. **(M.U. 2001)**

$$\left[\textbf{Ans. :}\ f(x) = -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n\sin n\pi x}{n^3}\right]$$

8. $f(x) = x^2 - 2, \quad -2 \le x \le 2$ **(M.U. 2000, 04, 05)**

$$\left[\textbf{Ans. :}\ f(x) = -\frac{2}{3} - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{2} - \frac{1}{4}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2}\ldots\right]\right]$$

9. $f(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ **(M.U. 1996)**

$$\left[\textbf{Ans. :}\ f(x) = \frac{k}{2} + \frac{2k}{\pi}\left[\frac{1}{1}\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{2\pi x}{2} + \frac{1}{5}\cos\frac{5x}{2} - \ldots\right]\right]$$

10. $f(x) = \begin{cases} a(x - l), & -l < x < 0 \\ a(l + x), & 0 < x < l \end{cases}$

Hence, deduce that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$

$$\left[\textbf{Ans.:}\ f(x) = \frac{2al}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 - 2(-1)^n\right]\sin\frac{n\pi x}{l}\right.$$

Then put $x = l/2$.]

11. $f(x) = \begin{cases} 0, & -2 < x < -1 \\ 1 + x, & -1 < x < 0 \\ 1 - x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ **(M.U. 2002)**

$$\left[\text{Ans. :}\ f(x) = \frac{1}{4} + \sum\frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right)\right]$$

12. $f(x) = e^{-x}\,(-a, a)$. **(M.U. 2003)**

$$\left[\textbf{Ans. :}\ f(x) = \frac{\sinh a}{a} + 2a \sinh a \sum \frac{(-1)^n}{a^2 + n^2\pi^2} \cos \frac{n\pi x}{a}\right.$$

$$\left. + 2\pi \sinh a \sum \frac{(-1)^{n+1} \cdot n}{a^2 + n^2\pi^2} \sin \frac{n\pi x}{a}\right]$$

7. Half Range Series

We have seen in § 6 that in $(-l, l)$, $f(x)$ can be expanded in Fourier Series as

$$f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right) + \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \dfrac{1}{2l}\displaystyle\int_{-l}^{l} f(x)\,dx,\ a_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$ and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Further by the properties of even function if $f(x)$ is even in $(-l, l)$ then

$$a_0 = \frac{1}{2l}\cdot 2\int_0^l f(x)\,dx = \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{1}{l}\cdot 2\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx \quad \text{and } b_n = 0$$

Thus, if we imagine an even function in $(-l, l)$ *i.e.* if we define $f(x)$ in $(0, l)$ and take its reflection in the y-axis so that $f(x)$ becomes even in $(-l, l)$ then Fourier Series of $f(x)$ is given by

$$f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$$

where, $$\boxed{\begin{aligned} a_0 &= \frac{1}{l}\int_0^l f(x)\,dx \\ a_n &= \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx \quad \text{and } b_n = 0\end{aligned}}$$

Such a series is called **half range cosine series.** Clearly this expansion does not contain sine terms.

Similarly, if we define $f(x)$ in $(0, l)$ and take its reflection in the x-axis so that $f(x)$ becomes odd in $(-l, l)$ then by the properties of odd function, $f(x)$ is given by

$$f(x) = \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

where, $$\boxed{b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \quad a_0 = 0 \ \text{and} \ a_n = 0}$$

Such a series is called **half-range sine series**. Clearly this expansion does not contain cosine terms.

Ex. 1 : Find a cosine series of period 2π to represent $\sin x$ in $0 \le x \le \pi$. **(M.U. 1995, 99, 2003, 04, 05)**

Sol. : Let $f(x) = a_0 + \sum a_n \cos nx \qquad [\because\ l = \pi]$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = \frac{1}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx \qquad \text{.......... (1)}$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(1+n)x + \sin(1-n)x]\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$a_n = 0$ if n is odd and $n \neq 1$. And

$$a_n = \frac{-2}{\pi}\left[-\frac{1}{n+1} + \frac{1}{n-1}\right] \quad \text{if } n \text{ is even}$$

$$= -\frac{4}{\pi}\left(\frac{1}{n^2-1}\right) \quad \text{if } n \text{ is even}$$

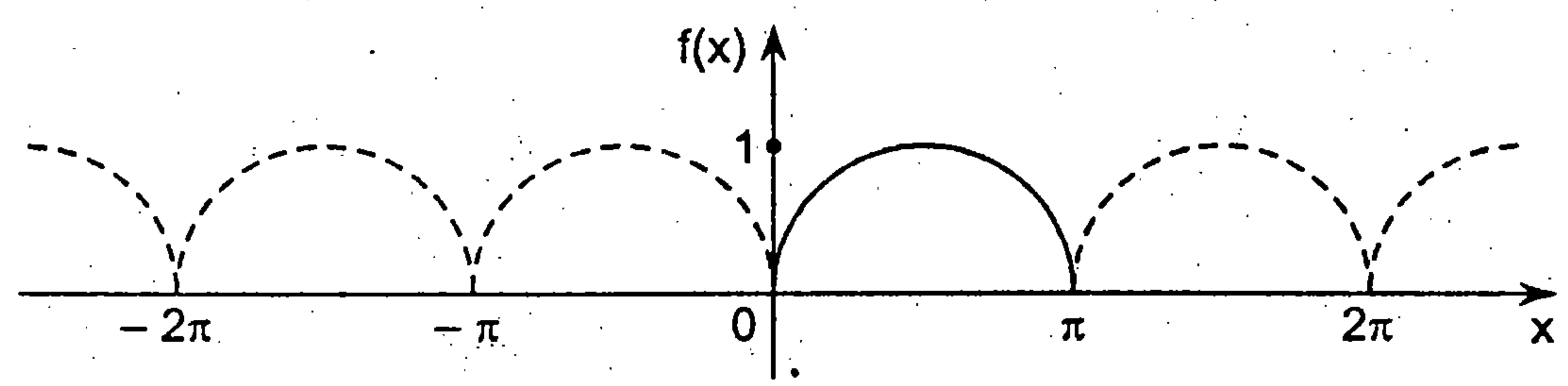
$\therefore$ If $n = 1$ from (1), we get

$$a_1 = \frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = 0$$

$$\therefore \quad f(x) = \sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots\right) \qquad \text{.......... (2)}$$

The graph of the function is given below.



Cor. 1 : Hence, deduce that

$$\frac{1}{1^2\cdot 3^2}+\frac{1}{3^2\cdot 5^2}+\frac{1}{5^2\cdot 7^2}+\ldots=\frac{\pi^2-8}{16} \qquad \textbf{(M.U. 1996)}$$

Sol. : By Paresval's identity (Cor. 6, page 7-8), we have

$$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2\,dx=\frac{1}{2}\left[2a_0^2+a_1^2+a_2^2+a_3^2+\ldots\right]$$

$$\therefore\quad \frac{1}{\pi}\int_0^{\pi}\sin^2 x\,dx=\frac{1}{2}\left[2a_0^2+a_1^2+a_2^2+a_3^2+\ldots\right]$$

Now, $\displaystyle \frac{1}{\pi}\int_0^{\pi}\sin^2 x\,dx=\frac{1}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx$

$$=\frac{1}{2\pi}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}=\frac{1}{2\pi}[\pi]=\frac{1}{2}$$

$$\therefore\quad \frac{1}{2}=\frac{1}{2}\left[2\cdot\frac{4}{\pi^2}+\frac{16}{\pi^2}\left\{\frac{1}{3^2}+\frac{1}{15^2}+\frac{1}{35^2}+\ldots\right\}\right]$$

$$1=\frac{8}{\pi^2}+\frac{16}{\pi^2}\left[\frac{1}{1^2\cdot 3^2}+\frac{1}{3^2\cdot 5^2}+\frac{1}{5^2\cdot 7^2}+\ldots\right]$$

$$1-\frac{8}{\pi^2}=\frac{16}{\pi^2}\left[\frac{1}{1^2\cdot 3^2}+\frac{1}{3^2\cdot 5^2}+\frac{1}{5^2\cdot 7^2}+\ldots\right]$$

$$\frac{\pi^2-8}{16}=\frac{1}{1^2\cdot 3^2}+\frac{1}{3^2\cdot 5^2}+\frac{1}{5^2\cdot 7^2}+\ldots$$

Cor. 2 : From the above expansion deduce that

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots \qquad \textbf{(M.U. 2005)}$$

Sol. : The series (2) can be written as

$$\sin x=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n)^2-1}\cos 2nx$$

Now, putting $x=\dfrac{\pi}{2}$, we get

$$1=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\cos n\pi$$

$$\frac{\pi}{2}=1-2\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)(-1)^n$$

$$=1-2\left[\frac{1}{2}\left\{-\left(\frac{1}{1}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)-\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)-\ldots\right\}\right]$$

$$=1-\left[-\left(\frac{1}{1}+\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)-\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\ldots\right]$$

$$=1+\left[1-\frac{1}{3}-\frac{1}{3}+\frac{1}{5}+\frac{1}{5}-\frac{1}{7}-\frac{1}{7}+\ldots\right]=2\cdot\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right]$$

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

Cor. 3 : From the above expansion deduce that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots=\frac{1}{2} \qquad \textbf{(M.U. 1997)}$$

Sol. : Putting $x=0$ in (2) we get,

$$-\frac{2}{\pi}=-\frac{4}{\pi}\left[\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\ldots\right]$$

$$\therefore\quad \frac{1}{2}=\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots$$

Ex. 2 : Obtain half range sine series for $f(x)$ when

$$f(x)=\begin{cases} x, & 0<x<(\pi/2)\\ \pi-x, & (\pi/2)<x<\pi\end{cases}$$

Hence, find the sum of $\displaystyle\sum_{(2n-1)}^{\infty}\frac{1}{n^4}$. **(M.U. 1993, 2002, 03)**

Sol. : Let $f(x)=\sum b_n\sin nx \qquad [\because\ l=\pi]$

$$\therefore\quad b_n=\frac{2}{\pi}\left[\int_0^{\pi/2}x\sin nx\,dx+\int_{\pi/2}^{\pi}(\pi-x)\sin n\,dx\right]$$

$$=\frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right)-\left(-\frac{\sin nx}{n^2}\right)(1)\right\}_0^{\pi/2}\right.$$

$$\left.+\left\{(\pi-x)\left(-\frac{\cos nx}{n}\right)-\left(-\frac{\sin nx}{n^2}\right)(-1)\right\}_{\pi/2}^{\pi}\right]$$

$$=\frac{2}{\pi}\left[\left\{-\frac{\pi}{2}\frac{\cos(n\pi/2)}{n}+\frac{\sin(n\pi/2)}{n^2}-0-0\right\}\right.$$

$$\left.+\left\{0-0+\frac{\pi}{2}\frac{\cos(n\pi/2)}{n}+\frac{\sin(n\pi/2)}{n^2}\right\}\right]$$

$$=\frac{4}{\pi}\frac{\sin(n\pi/2)}{n^2}$$

$$\therefore \quad b_1 = \frac{4}{\pi}\cdot\frac{1}{1^2},\ b_2 = 0,\ b_3 = -\frac{4}{\pi}\cdot\frac{1}{3^2},\ b_4 = 0 \ldots\ldots$$

$$\therefore \quad f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + \ldots\ldots\right]$$

By Parseval's identity (Cor. 7, page 7-8)

$$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2\,dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots\ldots\infty\right]$$

$$\therefore \quad \frac{1}{\pi}\left[\int_0^{\pi/2} x^2 dx + \int_{\pi/2}^{\pi}(\pi - x)^2\,dx\right] = \frac{1}{2}\left[b_1^2 + b_2^2 + \ldots\ldots\infty\right]$$

Now, $\frac{1}{\pi}\left[\int_0^{\pi/2} x^2\,dx + \int_{\pi/2}^{\pi}(\pi^2 - 2\pi x + x^2)\,dx\right]$

$$= \frac{1}{\pi}\left[\left\{\frac{x^3}{3}\right\}_0^{\pi/2} + \left\{\pi^2 x - \pi x^2 + \frac{x^3}{3}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi^3}{24} - 0\right\} + \left\{\left(\pi^3 - \pi^3 + \frac{\pi^3}{3}\right) - \left(\frac{\pi^3}{2} - \frac{\pi^3}{4} + \frac{\pi^3}{24}\right)\right\}\right]$$

$$= \frac{1}{\pi}\cdot\frac{\pi^3}{12} = \frac{\pi^2}{12}$$

$$\therefore \quad \frac{\pi^2}{12} = \frac{1}{2}\cdot\left[\frac{16}{\pi^2}\cdot\frac{1}{1^4} + \frac{16}{\pi^2}\cdot\frac{1}{3^4} + \frac{16}{\pi^2}\cdot\frac{1}{5^4} + \ldots\right]$$

$$\therefore \quad \frac{\pi}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$$

Ex. 3 : Find half range cosine series for $f(x) = x,\ 0 < x < 2$. Using Parseval's identity, deduce that

(i) $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$ **(M.U. 1994, 96, 2001, 02, 05)**

(ii) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$

Sol. : Let $f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$. Here, $l = 2$.

$$\therefore \quad a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left[x\frac{\sin(n\pi x/2)}{n\pi/2} + \frac{\cos(n\pi x/2)}{n^2\pi^2/2^2}\cdot 1\right]_0^2$$

$$= \left[2\cdot(0) + \frac{\cos n\pi}{n^2\pi^2/2^2} - 0 - \frac{1}{n^2\pi^2/2^2}\right] = \frac{[(-1)^n - 1]}{n^2\pi^2/2^2}$$

$$= \begin{cases} -4\cdot\frac{2}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \ldots\right]$$

(i) By Parseval's identity (Cor. 8, page 7-8)

$$\frac{1}{l}\int_0^l [f(x)]^2\,dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + \ldots\right]$$

$$\therefore \quad \text{l.h.s.} = \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$$

$$\therefore \quad \frac{4}{3} = \frac{1}{2}\left[2 + \frac{64}{\pi^4}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right\}\right]$$

$$\frac{8}{3} - 2 = \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right]$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$$

(ii) Let $S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots\right)$$

$$= \left(\frac{\pi^4}{96}\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right)$$

$$\therefore \quad S = \frac{\pi^4}{96} + \frac{S}{16} \qquad \therefore \quad S = \frac{\pi^4}{90}.$$

Ex. 4 : Find half range sine series for $x\sin x$ in $(0, \pi)$ and hence,

deduce that $\frac{\pi^2}{8\sqrt{2}} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \ldots$

Sol. : Let $f(x) = \sum b_n \sin nx \qquad [\because l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \sin nx \, dx \qquad \text{..........(1)}$$

$$= -\frac{1}{\pi}\int_0^{\pi} x\left[\cos(n+1)x - \cos(n-1)x\right] dx$$

$$= -\frac{1}{\pi}\left[x\left\{\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\cos(n+1)x}{(n+1)^2} + \frac{\cos(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\pi\left\{\frac{\sin(n+1)\pi}{n+1} - \frac{\sin(n-1)\pi}{n-1}\right\} - \left\{-\frac{\cos(n+1)\pi}{(n+1)^2} + \frac{\cos(n-1)\pi}{(n-1)^2}\right\} - 0 - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right] \text{ if } n \neq 1$$

Now, $\sin(n \pm 1)\pi = 0$ and $\cos(n \pm 1)\pi = -\cos n\pi$.

$$= -\frac{1}{\pi}\left[-\frac{\cos n\pi}{(n+1)^2} + \frac{\cos n\pi}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right]$$

$$= -\frac{1}{\pi}\begin{cases} 0 & \text{when } n \text{ is odd and } n \neq 1 \\ 2\left\{\dfrac{1}{(n-1)^2} - \dfrac{1}{(n+1)^2}\right\} & \text{when } n \text{ is even} \end{cases}$$

When $n = 1$ from (1),

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x \sin^2 x \, dx = \frac{2}{\pi}\int_0^{\pi} x \frac{(1-\cos 2x)}{2} dx$$

$$= \frac{1}{\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi^2 - 0 - \frac{\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{\pi}{2}$$

$$\therefore \quad f(x) = x \sin x = \frac{\pi}{2}\sin x - \frac{2}{\pi}\left[\left(\frac{1}{1^2} - \frac{1}{3^2}\right)\sin 2x + \left(\frac{1}{3^2} - \frac{1}{5^2}\right)\sin 4x + \left(\frac{1}{5^2} - \frac{1}{7^2}\right)\sin 6x + \ldots\right]$$

$$= \frac{\pi}{2}\sin x + \frac{2}{\pi}\left[\left(\frac{1}{3^2} - \frac{1}{1^2}\right)\sin 2x + \left(\frac{1}{5^2} - \frac{1}{3^2}\right)\sin 4x + \left(\frac{1}{7^2} - \frac{1}{5^2}\right)\sin 6x + \ldots\right]$$

For deduction put $x = \frac{\pi}{4}$.

$$\therefore \quad \frac{\pi}{4}\sin\frac{\pi}{4} = \frac{\pi}{2}\frac{1}{\sqrt{2}} + \frac{2}{\pi}\left[\left(\frac{1}{3^2} - \frac{1}{1^2}\right)\cdot 1 + \left(\frac{1}{5^2} - \frac{1}{3^2}\right)\cdot 0 + \left(\frac{1}{7^2} - \frac{1}{5^2}\right)\cdot(-1) + \ldots\right]$$

$$\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}} = \frac{\pi}{2}\cdot\frac{1}{\sqrt{2}} + \frac{2}{\pi}\left[\frac{1}{3^2} - \frac{1}{1^2} + \frac{1}{7^2} - \frac{1}{5^2} + \ldots\right]$$

$$-\frac{\pi}{4\sqrt{2}} = \frac{2}{\pi}\left[\frac{1}{3^2} - \frac{1}{1^2} + \frac{1}{7^2} - \frac{1}{5^2} + \ldots\right]$$

$$\frac{\pi^2}{8\sqrt{2}} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$$

Ex. 5 : In $(0, \pi)$ show that

$$x^2 = \frac{2}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x \ldots\right]$$

Sol. : Let $f(x) = \sum b_n \sin nx \quad [\because \; l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx \, dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{n}(-\cos n\pi) - 0 + \frac{2}{n^3}(\cos n\pi) - \frac{2}{n^3}\right]$$

$$= \begin{cases} \dfrac{2}{\pi}\left[\dfrac{\pi^2}{n} - \dfrac{4}{n^3}\right] & \text{if } n \text{ is odd} \\ \dfrac{2}{\pi}\left[-\dfrac{\pi^2}{n}\right] & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad x^2 = \frac{2}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x \ldots\right]$$

Ex. 6 : Expand $f(x) = \begin{cases} kx & 0 < x < l/2 \\ 0 & l/2 < x < l \end{cases}$ into half range cosine series.

Deduce the sum of the series $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$ **(M.U. 1998)**

Sol. : Let $f(x) = a_0 + \sum a_n \cos\left(\dfrac{n\pi x}{l}\right)$

$$\therefore \quad a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^{l} k(l-x)\,dx\right]$$

$$= \frac{1}{l}\left\{\left[k\frac{x^2}{2}\right]_0^{l/2} + \left[k\,l\,x - k\frac{x^2}{2}\right]_{l/2}^{l}\right\} = \frac{k\,l}{4}$$

$$a_n = \frac{2}{l}\left[\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} k(l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left\{kx\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - k\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right\}_0^{l/2}\right.$$

$$\left. + \left\{k(l-x)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (-k)\left(\frac{-l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right\}_{l/2}^{l}\right]$$

$$= \frac{2}{l}\left[\frac{k\,l^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{k\,l^2}{n^2\pi^2} - \frac{k\,l^2}{n^2\pi^2}\cos n\pi + \frac{k\,l^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$a_1 = 0,\ a_2 = -\frac{8\,k\,l}{2^2\pi^2},\ a_3 = 0,\ a_4 = 0,\ a_5 = 0$$

$$a_6 = -\frac{8\,k\,l}{6^2\pi^2},\ \ldots\ldots a_{10} = -\frac{8\,k\,l}{10^2\pi^2}$$

$$\therefore \quad f(x) = \frac{k\,l}{4} - \frac{8k\,l}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \ldots\right]$$

Now, put $x = \dfrac{l}{2}$, $\quad \therefore \quad \dfrac{k\,l}{2} - \dfrac{k\,l}{4} = \dfrac{8k\,l}{\pi^2}\left[\dfrac{1}{2^2} + \dfrac{1}{6^2} + \dfrac{1}{10^2} + \ldots\right]$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Ex. 7 : Find half range cosine series for

$$f(x) = 1, \quad 0 < x < \frac{a}{2}$$
$$= -1, \quad \frac{a}{2} < x < a$$

Sol. : Let $f(x) = a_0 + \sum a_n \cos\dfrac{n\pi x}{a} \qquad [\because\ l = a]$

$$a_0 = \frac{1}{a}\left[\int_0^{a/2} dx - \int_{a/2}^{a} dx\right]$$

$$= \frac{1}{a} + \left[\{x\}_0^{a/2} - \{x\}_{a/2}^{a}\right] = \frac{1}{a}\left[\frac{a}{2} - a + \frac{a}{2}\right] = 0$$

$$a_n = \frac{2}{a}\left[\int_0^{a/2} \cos\frac{n\pi x}{a}\,dx - \int_{a/2}^{a} \cos\frac{n\pi x}{a}\,dx\right]$$

$$= \frac{2}{a}\left[\left\{\frac{a}{n\pi}\sin\frac{n\pi x}{a}\right\}_0^{a/2} - \left\{\frac{a}{n\pi}\sin\frac{n\pi x}{a}\right\}_{a/2}^{a}\right]$$

$$= \frac{2}{a}\left[\frac{a}{n\pi}\sin\frac{n\pi}{2} - \frac{a}{n\pi}\sin n\pi + \frac{a}{n\pi}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{n\pi}\left[2\sin\frac{n\pi}{2}\right]$$

$$\therefore \quad a_1 = \frac{4}{\pi},\quad a_3 = -\frac{4}{3\pi},\quad a_5 = \frac{4}{5\pi}$$

$$a_2 = 0,\quad a_4 = 0,\quad a_6 = 0\ \ldots\ldots$$

$$\therefore \quad f(x) = \frac{4}{\pi}\left[\frac{1}{1}\cos\frac{\pi x}{a} - \frac{1}{3}\cos\frac{\pi x}{a} + \frac{1}{5}\cos\frac{5\pi x}{a} \ldots\right]$$

Ex. 8 : If $f(x) = \dfrac{x}{a}$ for $0 < x < a$

$$= \frac{l-x}{l-a} \text{ for } a < x < l$$

prove that $f(x) = \dfrac{2\,l^2}{a(l-a)\pi^2}\sum \dfrac{1}{n^2}\sin\dfrac{n\pi a}{l}\sin\dfrac{n\pi x}{l}$

Sol. : Let $f(x) = \sum b_n \sin\dfrac{n\pi x}{l}$

$$b_n = \frac{2}{l}\left[\int_0^a \frac{x}{a}\sin\frac{n\pi x}{l}\,dx + \int_a^l \frac{l-x}{l-a}\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left\{\frac{x}{a}\cdot\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{1}{a}\right)\cdot\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right\}_0^a\right.$$

$$+ \left\{\left(\frac{l-x}{l-a}\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{-1}{l-a}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right\}_a^l$$

$$= \frac{2}{l}\left[-\frac{l}{n\pi}\cos\frac{n\pi a}{l} + \frac{l^2}{a\,n^2\pi^2}\sin\frac{n\pi a}{l}\right.$$

$$\left. + \frac{l}{n\pi}\cos\frac{n\pi a}{l} + \frac{l^2}{(l-a)\,n^2\pi^2}\sin\frac{n\pi a}{l}\right]$$

$$= \frac{2l}{n^2\pi^2}\sin\frac{n\pi a}{l}\left[\frac{1}{a} + \frac{1}{l-a}\right] = \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi a}{l}\left(\frac{1}{a(l-a)}\right)$$

$$\therefore \quad f(x) = \frac{2l^2}{a(l-a)\pi^2}\sum \frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

Ex. 9 : Obtain the expansion of $f(x) = x(\pi - x)$, $0 < x < \pi$ as a half-range cosine series. Hence, show that

$$\text{(i)}\ \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \text{(ii)}\ \sum_1^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12},$$

$$\text{(iii)}\ \sum_1^\infty \frac{1}{n^4} = \frac{\pi^4}{90} \qquad \textbf{(M.U. 1993, 96, 99, 2005)}$$

Sol. : Let $f(x) = a_0 + \sum a_n \cos nx \qquad [\because\ l = \pi]$

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\int_0^\pi (\pi x - x^2)\,dx = \frac{1}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^\pi = \frac{\pi^2}{6}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\cdot\frac{\sin nx}{n} - (\pi - 2x)\cdot\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^\pi$$

$$= -2\left(\frac{1 + \cos n\pi}{n^2}\right) = \begin{cases} 0 & \text{when } n \text{ is odd} \\ -4/n^2 & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore \quad x(\pi - x) = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \ldots\right]$$

$$= \frac{\pi^2}{6} - \left[\frac{1}{1^2}\cos 2x + \frac{1}{2^2}\cos 4x + \frac{1}{3^2}\cos 6x + \ldots\right]$$

(i) Now, put $x = 0$

$$\therefore \quad 0 = \frac{\pi^2}{6} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$\therefore \quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

(ii) Again put $x = \frac{\pi}{2}$

$$\therefore \quad \frac{\pi^2}{4} = \frac{\pi^2}{6} - \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\right]$$

$$\therefore \quad \frac{\pi^2}{4} - \frac{\pi^2}{6} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

(iii) Now, by Parseval's identity (Cor. 6, page 7-8).

$$\frac{1}{\pi}\int_0^\pi [f(x)]^2\,dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + a_3^2 + \ldots\right]$$

$$\therefore \quad \frac{1}{\pi}\int_0^\pi x^2(\pi - x)^2\,dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + a_3^2 + \ldots\right]$$

Now, $\quad \frac{1}{\pi}\int_0^\pi x^2(\pi - x)^2\,dx = \frac{1}{\pi}\int_0^\pi (\pi^2 x^2 - 2\pi x^3 + x^4)\,dx$

$$= \frac{1}{\pi}\left[\pi^2\frac{x^3}{3} - 2\pi\frac{x^4}{4} + \frac{x^5}{5}\right]_0^\pi = \frac{1}{\pi}\cdot\frac{\pi^5}{30} = \frac{\pi^4}{30}$$

$$\therefore \quad \frac{\pi^4}{30} = \frac{1}{2}\left[2\cdot\frac{\pi^4}{36} + \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right]$$

$$\frac{\pi^4}{15} - \frac{\pi^4}{18} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$$

Ex. 10 : Show that if $0 < x < \pi$

$$\cos x = \frac{8}{\pi}\sum_{m=1}^\infty \frac{m}{4m^2 - 1}\sin 2mx. \qquad \textbf{(M.U. 2002, 03, 04)}$$

Sol. : We have to obtain half-range sine series for $\cos x$

Let $\cos x = \sum b_n \sin nx \qquad [\because\ l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi \cos x \sin nx\,dx \qquad \ldots\ldots\ldots (1)$$

$$= \frac{1}{\pi}\int_0^\pi [\sin(1+n)x - \sin(1-n)x]\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} + \frac{\cos \pi x}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1} + \frac{1}{n-1}\right](1+\cos n\pi) = \frac{1}{\pi}\cdot\frac{2n}{n^2-1}[1+(-1)^n]$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd and } n \neq 1 \\ \frac{1}{\pi}\cdot\frac{4n}{n^2-1}, & \text{is } n \text{ is even} \end{cases}$$

When $n = 1$, from (1), we get,

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\cos x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x \, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{2\pi}[1-1] = 0$$

$$\therefore \quad \cos x = \frac{4}{\pi}\left[\frac{2}{2^2-1}\sin 2x + \frac{4}{4^2-1}\sin 4x + \frac{6}{6^2-1}\sin 6x + \ldots\right]$$

$$= \frac{8}{\pi}\left[\frac{1}{2^2-1}\sin 2x + \frac{2}{4^2-1}\sin 4x + \frac{3}{6^2-1}\sin 6x + \ldots\right]$$

$$= \frac{8}{\pi}\sum\frac{m}{4m^2-1}\sin 2m\pi.$$

Ex. 11 : Expand $f(x) = l\,x - x^2$. $0 < x < l$ in a half-range

(i) cosine series, (ii) sine series.

Hence, from sine series deduce that

(i) $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$ **(M.U. 1994, 2003, 05)**

(ii) $\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \ldots$

(iii) $\frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots$

Sol. : (i) Cosine Series

Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l}$

$$a_0 = \frac{1}{l}\int_0^l (l\,x - x^2)\,dx = \frac{1}{l}\left[l\frac{x^2}{2} - \frac{x^3}{3}\right]_0^l = \frac{1}{l}\cdot\frac{l^3}{6} = \frac{l^2}{6}$$

$$a_n = \frac{2}{l}\int_0^l (l\,x - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(l\,x - x^2)\left(\frac{1}{n\pi}\sin\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{0 - l\cdot\frac{l^2}{n^2\pi^2}\cos n\pi - 0\right\} - \left\{0 + l\cdot\frac{l^2}{n^2\pi^2} + 0\right\}\right]$$

$$= -\frac{2}{l}\cdot\frac{l^3}{n^2\pi^2}[\cos n\pi + 1]$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd} \\ -\frac{4\,l^2}{n^2\pi^2}, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad f(x) = l\,x - x^2 = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{4^2}\cos\frac{4\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \ldots\right]$$

(ii) Sine series

Let $f(x) = \sum b_n \sin\frac{n\pi x}{l}$

$$b_n = \frac{2}{l}\int_0^l (l\,x - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(l\,x - x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{0 - 0 - \frac{2\,l^3}{n^3\pi^3}\cos n\pi\right\} - \left\{0 - 0 - \frac{2\,l^3}{n^3\pi^3}\right\}\right]$$

$$= \frac{2}{l}\cdot\frac{2\,l^3}{n^3\pi^3}(-\cos n\pi + 1)$$

$$= \begin{cases} \dfrac{8l^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad f(x) = lx - x^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \ldots\right]$$

(i) For deduction we put $x = \dfrac{l}{2}$

$$\therefore \quad l \cdot \left(\frac{l}{2}\right) - \left(\frac{l}{2}\right)^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right]$$

$$\therefore \quad \frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$$

(ii) By corollary 9, page 1-8

$$\frac{1}{l}\int_0^l [f(x)]^2\, dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots \infty\right]$$

$$\therefore \quad \frac{1}{l}\int_0^l [lx - x^2]^2\, dx = \frac{1}{2}\left(\frac{8l^2}{\pi^3}\right)^2\left[\left(\frac{1}{1^3}\right)^2 + \left(\frac{1}{3^3}\right)^2 + \left(\frac{1}{5^3}\right)^2 + \ldots\right]$$

$$\frac{1}{l}\int_0^l (l^2x^2 - 2lx^3 + x^4)\, dx = \frac{32l^4}{\pi^6}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right]$$

$$\frac{1}{l}\left[l^2\frac{x^3}{3} - 2l\frac{x^4}{4} + \frac{x^5}{5}\right]_0^l = \frac{32l^4}{\pi^6}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right]$$

$$\therefore \quad \frac{1}{30}l^4 = \frac{32l^4}{\pi^6}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right]$$

$$\therefore \quad \frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots$$

(iii) Let $S = \dfrac{1}{1^6} + \dfrac{1}{2^6} + \dfrac{1}{3^6} + \dfrac{1}{4^6} + \ldots$

$$= \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \ldots\right)$$

$$\therefore \quad S = \frac{\pi^6}{960} + \frac{1}{2^6}\left(\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} \ldots\right)$$

$$= \frac{\pi^6}{960} + \frac{1}{64}S.$$

$$\therefore \quad S - \frac{S}{64} = \frac{\pi^6}{960} \qquad \therefore \quad \frac{63}{64}S = \frac{\pi^6}{960}$$

$$\therefore \quad S = \frac{\pi^6}{960} \cdot \frac{64}{63} = \frac{\pi^6}{945}$$

$$\therefore \quad \frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots$$

Ex. 12 : Find half range sine series for $f(x)$ where

$$f(x) = \begin{cases} x, & 0 < x \le (\pi/2) \\ \pi - x, & (\pi/2) < x < \pi \end{cases}$$

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$ **(M.U. 1993)**

Sol. : Let $f(x) = \sum b_n \sin nx \qquad [\because\ l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}(x)\sin nx\, dx + \int_{\pi/2}^\pi (\pi - x)\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin x}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left\{-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right\} + \left\{\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$= \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

$$\therefore \quad b_1 = \frac{4}{\pi 1^2}, \quad b_3 = -\frac{4}{\pi 3^2}, \quad b_5 = \frac{4}{\pi 5^2}, \ \ldots\ldots$$

$$b_2 = 0, \quad b_4 = 0, \quad b_6 = 0 \ \ldots\ldots$$

Hence, $f(x) = \dfrac{4}{\pi}\left[\dfrac{1}{1^2}\sin x - \dfrac{1}{3^2}\sin 3x + \dfrac{1}{5^2}\sin 5x - \ldots\right]$

Now, put $x = \dfrac{\pi}{2}$

$$\therefore \quad \frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Ex. 13 : Find half range cosine series for

$$f(x) = \begin{cases} x, & 0 < x < (\pi/2) \\ \pi - x, & (\pi/2) < x < \pi \end{cases}$$ **(M.U. 2002)**

Sol. : Let $f(x) = a_0 + \sum a_n \cos nx \qquad [\because\ l = \pi]$

$$\therefore \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{x^2}{2}\right\}_0^{\pi/2} + \left\{\pi x - \frac{x^2}{2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right] = \frac{\pi}{4}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\cdot\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{\pi}{2}\cdot\frac{1}{n}\cdot\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right\} + \left\{-\frac{1}{n^2}\cos n\pi - \frac{\pi}{2}\cdot\frac{1}{n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right\}\right]$$

$$= \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right]$$

$$\therefore \quad a_1 = 0,\ a_2 = \frac{2}{\pi\cdot 2^2}[2(-1) - (+1) - 1] = -\frac{8}{\pi\cdot 2^2}$$

$$a_3 = 0,\ a_4 = 0,\ a_5 = 0$$

$$a_6 = \frac{2}{\pi\cdot 6^2}[2(-1) - (+1) - 1] = -\frac{8}{\pi\cdot 6^2}$$

$$a_7 = 0,\ a_8 = 0,\ a_9 = 0$$

$$a_{10} = \frac{2}{\pi\cdot 10^2}[2(-1) - (+1) - 1] = -\frac{8}{\pi\cdot 10^2}$$

$$\therefore \quad f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left[\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{10^2}\cos 10x + \ldots\right]$$

Ex. 14 : Obtain half range sine series to represent

$$f(x) = \begin{cases} \dfrac{2x}{3}, & 0 \le x \le \dfrac{\pi}{3} \\ \dfrac{\pi - x}{3}, & \dfrac{\pi}{3} \le x \le \pi \end{cases}$$

Sol. : Let $f(x) = \sum b_n \sin n\pi x \qquad [\because\ l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin n\pi x\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/3}\frac{2x}{3}\sin n\pi x\,dx + \int_{\pi/3}^{\pi}\left(\frac{\pi - x}{3}\right)\sin n\pi x\,dx\right]$$

$$= \frac{2}{3\pi}\left[\int_0^{\pi/3} 2x\sin n\pi x\,dx + \int_{\pi/3}^{\pi}(\pi - x)\sin n\pi x\,dx\right]$$

$$= \frac{2}{3\pi}\left[\left\{(2x)\left(-\frac{\cos n\pi x}{n}\right) - (2)\left(-\frac{\sin n\pi x}{n^2}\right)\right\}_0^{\pi/3} + \left\{(\pi - x)\left(-\frac{\cos n\pi x}{n}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2}\right)\right\}_{\pi/3}^{\pi}\right]$$

$$= \frac{2}{3\pi}\left[\left\{\left(\frac{2\pi}{3}\right)\left(-\frac{1}{n}\cos\frac{n\pi}{3}\right) + \frac{2}{n^2}\sin\frac{n\pi}{3}\right\} + \left\{0 + \frac{2\pi}{3}\left(\frac{1}{n}\cos\frac{n\pi}{3}\right) + \frac{1}{n^2}\sin\frac{n\pi}{3}\right\}\right]$$

$$= \frac{2}{3\pi}\cdot\frac{3}{n^2}\sin\frac{n\pi}{3} = \frac{2}{\pi}\cdot\frac{1}{n^2}\sin\frac{n\pi}{3}$$

$$\therefore \quad f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin nx$$

$$= \frac{2}{\pi}\left[\frac{1}{1^2}\cdot\frac{\sqrt{3}}{2}\sin x + \frac{1}{2^2}\frac{\sqrt{3}}{2}\cdot\cos 2x - \frac{1}{4^2}\cdot\frac{\sqrt{3}}{2}\sin 4x - \frac{1}{5^2}\cdot\frac{\sqrt{3}}{2}\sin 5x + \ldots\right]$$

$$= \frac{\sqrt{3}}{\pi}\left[\frac{1}{1^2}\sin x + \frac{1}{2^2}\sin 2x - \frac{1}{4^2}\sin 4x - \frac{1}{5^2}\sin 5x + \ldots\right]$$

Ex. 15 : Obtain half-range sine series in $(0, \pi)$ for $x(\pi - x)$ and hence, find the value of $\sum \frac{(-1)^n}{(2n+1)^3}$. **(M.U. 1996)**

Sol. : Let $f(x) = \sum b_n \sin nx$

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[x(\pi - x)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 - 0 - \frac{2}{n^3}\cos(n\pi)\right\} - \left\{0 - 0 - \frac{2}{n^3}\right\}\right]$$

$$= \frac{4}{\pi}\left[\frac{1 - \cos n\pi}{n^3}\right] = \frac{4}{\pi n^3}\left[1 - (-1)^n\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{8}{\pi n^3} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad x(\pi - x) = \frac{8}{\pi}\left[\frac{1}{1^3}\sin \pi x + \frac{1}{3^3}\sin 3\pi x + \frac{1}{5^3}\sin 5x + \ldots\right]$$

Now, put $x = \frac{\pi}{2}$

$$\therefore \quad \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots\right]$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$$

Ex. 16 : Find half-range sine series for $f(x) = \frac{\pi}{4}$ in $(0, \pi)$. Hence, show that

(i) $\frac{\pi}{4}\left(\frac{\pi}{2} - x\right) = \frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots$

(ii) $\frac{\pi}{8}x(\pi - x) = \frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \ldots$

(iii) $\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$

Sol. : Let $f(x) = \sum b_n \sin nx$

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}\frac{\pi}{4}\sin nx\, dx$$

$$= \frac{1}{2}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{1}{2n}\left[\cos n\pi - 1\right]$$

$$= \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad \frac{\pi}{4} = \frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots \qquad \ldots\ldots\ldots (1)$$

(i) Now, integrate both sides of (1) w.r.t. x from 0 to x.

$$\therefore \quad \frac{\pi}{4}\int_0^x dx = \frac{1}{1}\int_0^x \sin dx + \frac{1}{3}\int_0^x \sin 3x\, dx + \ldots$$

$$\frac{\pi}{4}[x]_0^x = \frac{1}{1}[-\cos x]_0^x + \frac{1}{3}\left[-\frac{\cos 3x}{3}\right]_0^x + \ldots$$

$$\frac{\pi x}{4} = \frac{1}{1^2}(1 - \cos x) + \frac{1}{3^2}(1 - \cos 3x) + \ldots \qquad \ldots\ldots\ldots (2)$$

Now, put $x = \frac{\pi}{2}$ in (2)

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots \qquad \ldots\ldots\ldots (3)$$

Now, subtract (2) from (3)

$$\therefore \quad \frac{\pi^2}{8} - \frac{\pi x}{4} = \frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots$$

$$\therefore \quad \frac{\pi}{4}\left(\frac{\pi}{2} - x\right) = \frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots \qquad \ldots (4)$$

(ii) Now, integrate (4) w.r.t. x from 0 to x,

$$\frac{\pi}{4}\int_0^x\left(\frac{\pi}{2} - x\right)dx = \frac{1}{1^2}\int_0^x \cos x\, dx + \frac{1}{3^2}\int_0^x \cos 3x\, dx + \ldots$$

$$\frac{\pi}{4}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^x = \frac{1}{1^2}\left[\frac{\sin x}{1}\right]_0^x + \frac{1}{3^2}\left[\frac{\sin 3x}{3}\right]_0^x + \ldots$$

$$\therefore \quad \frac{\pi}{8}x(\pi - x) = \frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \ldots \qquad \ldots\ldots\ldots (5)$$

(iii) Now, put $x = \frac{\pi}{2}$ in (5),

$$\therefore \quad \frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots \qquad \ldots\ldots\ldots (6)$$

Ex. 17 : Prove that in the interval $0 < x < \pi$,

$$\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin x}{a^2+1} - \frac{2\sin 2x}{a^2+4} + \frac{3\sin 3x}{a^2+9} - \ldots\right]$$

(M.U. 1996, 2003, 05)

Sol. : Let $f(x) = e^{ax} - e^{-ax}$, $0 < x < \pi$, $\quad [\because l = \pi]$

Since, we want $f(x)$ to be expanded in sine terms only

Let $f(x) = \sum b_n \sin nx$

$$b_n = \int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}(e^{ax} - e^{-ax})\sin nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi} e^{ax}\sin nx\, dx - \int_0^{\pi} e^{-ax}\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right\}_0^{\pi} - \left\{\frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx)\right\}_0^{\pi}\right]$$

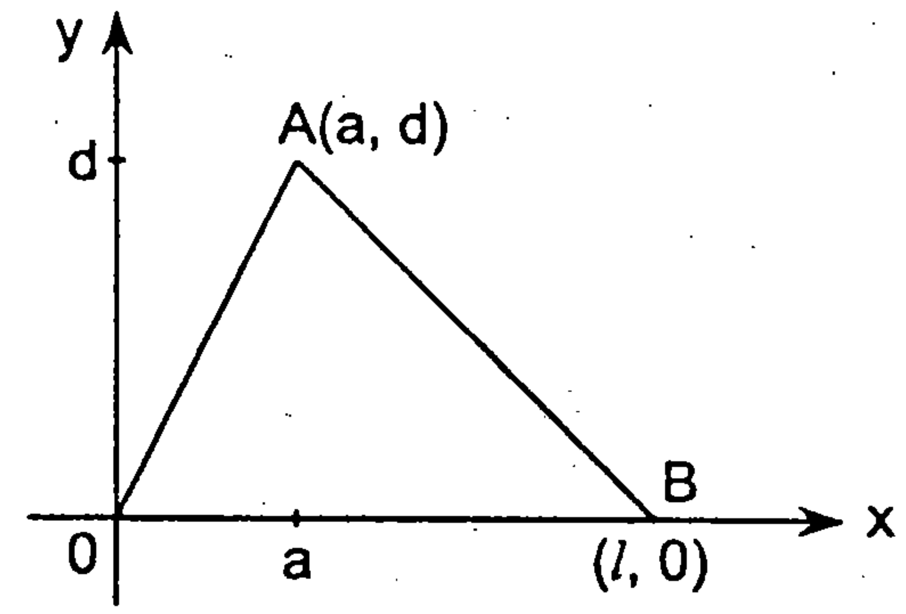
$$= \frac{2}{\pi}\left[\left\{\frac{e^{a\pi}}{a^2+n^2}(-n\cos n\pi) + \frac{n}{a^2+n^2}\right\} - \left\{\frac{e^{-a\pi}}{a^2+n^2}(-n\cos n\pi) + \frac{n}{a^2+n^2}\right\}\right]$$

$$= \frac{2}{\pi}\cdot\frac{e^{a\pi} - e^{-a\pi}}{a^2+n^2}(-n\cos n\pi)$$

$$\therefore \quad \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}(-n\cos n\pi)\sin nx$$

$$= \frac{2}{\pi}\left[\frac{1}{a^2+1}\sin x - \frac{2}{a^2+2^2}\sin 2x + \frac{3}{a^2+3^2}\sin 3x - \ldots\right]$$

Ex. 18 : Find half range sine series for $f(x)$ given by the following graph in the interval $(0, l)$.



Sol. : Since, A is (a, d). The equation of OA is $\frac{y-0}{0-d} = \frac{x-0}{0-a}$ $\quad \therefore \; y = \frac{d}{a}\cdot x$

Similarly, the equation of AB is $\frac{y-d}{d-0} = \frac{x-a}{a-l}$ $\quad \therefore \; y = d + \frac{d(x-a)}{a-l}$

$$\therefore \quad y = \frac{da - dl + dx - da}{a-l} = \frac{d}{a-l}(x-l)$$

$$\therefore \quad f(x) = \begin{cases} \frac{d}{a}\cdot x, & 0 \le x \le a \\ \frac{d}{a-l}(x-l), & a \le x \le l \end{cases}$$

Let $f(x) = \sum b_n \sin\frac{n\pi x}{l}$

Now, refer to the solved example 8 above where $d = 1$.

Following the same steps we can obtain.

$$f(x) = \frac{2dl^2}{\pi^2\cdot a\cdot(l-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\cdot\sin\frac{n\pi x}{l}$$

Ex. 19 : Find half-range cosine series for $f(x) = e^x$, $0 < x < 1$.

Sol. : Let $f(x) = a_0 + \sum a_n\cos n\pi x \quad [\because l = 1]$

$$\therefore \quad a_0 = \frac{1}{1}\int_0^1 e^x dx = \left[e^x\right]_0^1 = e - 1$$

$$a_n = \frac{2}{1}\int_0^1 e^x\cos n\pi x\, dx$$

$$= 2\left[\frac{e^x}{1+n^2\pi^2}(\cos n\pi x + n\pi\sin n\pi x)\right]_0^1$$

$$= 2\left[\frac{e}{1+n^2\pi^2}(-1)^n - \frac{1}{1+n^2\pi^2}\right]_0^1$$

$$\therefore \quad e^x = (e-1) + 2\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}\left[e(-1)^n - 1\right]\cos n\pi x$$

$$= (e-1) + 2\left[-\frac{1}{1+1^2 \cdot \pi^2}(e+1) \cdot \cos \pi x + \frac{1}{1+2^2 \cdot \pi^2}(e-1)\cos 2\pi x - \frac{1}{1+3^2 \cdot \pi^2}(e+1)\cos 3\pi x + \ldots\right]$$

EXERCISE

(A) 1. Obtain half range cosine series for

(i) $f(x) = x$ in $0 < x < 2$. **(M.U. 1994, 96)**

(ii) $f(x) = x$ in $0 < x < 1$.

$$\left[\textbf{Ans.: (i) } f(x) = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right]\right.$$

$$\left.\textbf{(ii) } f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos \pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5x + \ldots\right]\right]$$

2. Obtain cosine series for the function $(mx + c)$ in the interval $0 \le x \le p$ and hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

[**Ans. :** $f(x) = \left(\frac{mp}{2} + c\right) - \frac{4mp}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{p} + \frac{1}{3^2}\cos\frac{3\pi x}{p} + \frac{1}{5^2}\cos\frac{5\pi x}{p} + \ldots\right]$

Then put $x = 0$.

By putting $p = 2$, $m = 1$, $c = 0$, you get ex. 1(i), by putting $p = 1$, $m = 1$, $c = 0$ you get ex. 1(ii).]

3. $f(x) = \begin{cases} 1, & 0 < x < (c/2) \\ 1 - x, & (c/2) < x < c \end{cases}$

$$\left[\textbf{Ans. :} f(x) = \left(1 - \frac{3c}{8}\right) + \frac{2c}{\pi^2}\left[(\pi+1)\cos\frac{\pi x}{c} - \frac{1}{2}\cos\frac{2\pi x}{c} + \frac{1}{9}\left(1 - \frac{3\pi}{2}\right)\cos\frac{3\pi x}{c} + \ldots\right]\right]$$

4. Obtain half-range sine series for $f(x) = x(2 - x)$ in $0 < x < 2$ and hence, deduce that $\sum \frac{1}{n^6} = \frac{\pi^6}{945}$. **(M.U. 2003)**

5. Obtain half range sine series for

$$f(x) = \begin{cases} x, & 0 < x < a \\ a, & a < x < \pi - a \\ \pi - x, & \pi - a < x < \pi \end{cases}$$

$$\left[\textbf{Ans. :} f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\sin a \sin x + \frac{1}{3^2}\sin 3a \sin 3x + \frac{1}{5^2}\sin 5a \sin 5x + \ldots\right]\right]$$

6. Obtain half range sine series for

$$f(x) = \begin{cases} \pi/3, & 0 \le x < (\pi/3) \\ 0, & (\pi/3) \le x < (2\pi/3) \\ -\pi/3, & (2\pi/3) \le x \le \pi \end{cases}$$

$$\left[\textbf{Ans. :} f(x) = \sin 2x + \frac{1}{2}\sin 4x + \frac{1}{10}\sin 10x + \ldots\right]$$

7. Obtain half range cosine series for the above function.

$$\left[\textbf{Ans. :} f(x) = \frac{2}{\sqrt{3}}\left[\frac{1}{1}\cos x - \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x - \ldots\right]\right]$$

8. Obtain half range sine series for

$$f(x) = \begin{cases} (1/4) - x, & 0 < x < (1/2) \\ x - (3/4), & (1/2) < x < 1 \end{cases}$$ **(M.U. 1993, 2001)**

$$\left[\textbf{Ans. :} f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \ldots\right]$$

9. Show that in the interval $0 < x < \pi$,

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{1 \cdot 3} - \frac{\cos 3x}{2 \cdot 4} + \frac{\cos 4x}{3 \cdot 5} - \ldots\right]$$

(M.U. 1995, 97)

10. Find half range cosine series for

$$f(x) = a\left\{1 - \frac{x}{l}\right\}, \quad 0 < x < l.$$

$$\left[\textbf{Ans. :} f(x) = \frac{a}{2} + \frac{4a}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right]\right]$$

11. Find half range sine series for the above function.

$$\left[\textbf{Ans. :} f(x) = \frac{2a}{\pi}\left[\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\right]\right]$$

12. Obtain half-range cosine series for $f(x) = x$ in $0 < x < l$.

Hence, deduce that (i) $\dfrac{1}{1^4}+\dfrac{1}{3^4}+\dfrac{1}{5^4}+\ldots\ldots = \dfrac{\pi^4}{96}$

(ii) $\dfrac{1}{1^4}+\dfrac{1}{2^4}+\dfrac{1}{3^4}+\dfrac{1}{4^4}+\ldots\ldots = \dfrac{\pi^4}{90}$

(iii) $\dfrac{1}{2^4}+\dfrac{1}{4^4}+\dfrac{1}{6^4}+\ldots\ldots = \dfrac{\pi^4}{1440}$ **(M.U. 2001)**

13. Find half range sine series for

$$f(x) = \begin{cases} x, & 0 \le x \le (l/2) \\ l-x, & (l/2) \le x \le l \end{cases}$$

$$\left[\textbf{Ans. :}\ f(x) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cdot\sin\frac{n\pi}{2}\right)\cdot\sin\frac{n\pi x}{l}\right]$$

14. Obtain half-range cosine series for

$$f(x) = \begin{cases} kx, & 0 \le x \le (l/2) \\ k(l-x), & (l/2) \le x \le 1 \end{cases}$$

Hence, deduce that $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots\ldots = \dfrac{\pi^2}{8}$ **(M.U. 2001, 03)**

and $\dfrac{1}{1^4}+\dfrac{1}{3^4}+\dfrac{1}{5^4}+\ldots\ldots = \dfrac{\pi^4}{96}$. **(M.U. 2004)**

$$\left[\textbf{Ans. :}\ f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \ldots\ldots\right]\right.$$

Then, put $x = l/2$.]

15. Find half range cosine and sine series for $f(x)$, where

$$f(x) = \begin{cases} \pi/3, & 0 \le x \le (\pi/3) \\ 0, & (\pi/3) \le x \le (2\pi/3) \\ -\pi/3, & (2\pi/3) \le x \le \pi \end{cases}$$

$$\left[\textbf{Ans. : (i)}\ f(x) = \frac{2}{\sqrt{3}}\left[\frac{1}{1}\cos x - \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x - \ldots\right]\right.$$

$$\left.\textbf{(ii)}\ f(x) = \sin 2x + \frac{1}{2}\sin 4x + \frac{1}{10}\sin 10x + \ldots\right]$$

16. Find half range cosine series for $\cos\lambda x$ in $(0, \pi)$ where λ is not an integer and hence, show that $\pi\cot\pi\lambda = \dfrac{1}{\lambda} + \sum\limits_{n=1}^{\infty}\dfrac{2\lambda}{\lambda^2-n^2}$.

$$\left[\textbf{Ans. :}\ \cos\lambda x = \frac{\sin\lambda\pi}{\lambda\pi} + \frac{2\lambda\sin\lambda\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{\lambda^2-n^2}\cos nx\right]$$

17. Find half range sine series for $f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 4-x, & 2 \le x \le 4 \end{cases}$

(**Hint :** Put $l = 4$ in the above example number 11.)

$$\left[\textbf{Ans. :}\ f(x) = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\sin\frac{n\pi}{2}\right)\cdot\sin\frac{n\pi x}{4}\right]$$

18. Find half range sine series of period $2l$ for

$$f(x) = \frac{2x}{l},\quad 0 \le x \le \frac{l}{2}$$
$$= \frac{2}{l}(l-x),\quad \frac{l}{2} \le x \le l.$$

(M.U. 1998, 2002, 04)

$$\left[\textbf{Ans. :}\ f(x) = \frac{8}{\pi^2}\sum\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\right]$$

19. Find half range cosine series for

$$f(x) = 1,\quad 0 \le x \le 1$$
$$= x,\quad 1 \le x \le 2.$$

$$\left[\textbf{Ans. :}\ f(x) = \frac{5}{4} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right]\right]$$

20. Obtain half range cosine series for

$f(x) = x - x^2$ for $0 \le x \le 1$ **(M.U. 1994)**

(**Hint :** See solved example 11 above.)

$$\left[\textbf{Ans. :}\ f(x) = \frac{1}{6} - \frac{4}{\pi^2}\left[\frac{1}{2^2}\cos 2\pi x + \frac{1}{4^2}\cos 4\pi x + \ldots\right]\right]$$

21. Obtain half-range sine series for

$$f(x) = x,\quad 0 < x < 1$$
$$= 2-x,\quad 1 < x < 2.$$

Hence, deduce that $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dfrac{1}{7^2}+\ldots = \dfrac{\pi^2}{8}$

$$\left[\textbf{Ans. : } f(x) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} \ldots\right]\right]$$

22. Obtain half-range cosine series for

$f(x) = c - x, \quad 0 < x < c$

$$\left[\textbf{Ans. : } f(x) = \frac{c}{2} + \frac{4c}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{c} + \frac{1}{3^2}\cos\frac{3\pi x}{c} + \ldots\right]\right]$$

23. Obtain cosine series for

$f(x) = \frac{1}{2}(\pi - x)\sin x$ in $(0, \pi)$.

$$\left[\textbf{Ans. : } f(x) = \frac{1}{2} + \frac{1}{4}\cos x - \frac{1}{3}\cos 2x + \frac{1}{8}\cos 3x - \ldots\right]$$

24. Obtain half range sine series for

$f(x) = 1, \quad 0 < x < \frac{1}{2}$

$= 0, \quad \frac{1}{2} < x < 1.$ $\left[\textbf{Ans. : } f(x) = \frac{4}{\pi}\sum\frac{1}{n}\sin^2\frac{n\pi}{4}\sin n\pi x\right]$

25. Obtain half-range cosine series for $f(x) = x\sin x$ in $(0, \pi)$.

(M.U. 2004) $\left[\textbf{Ans. : } f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx\right]$

26. Obtain half range cosine series for

$f(x) = c, \quad 0 < x < a$

$= 0, \quad a < x < b.$

$$\left[\textbf{Ans. : } f(x) = \frac{ac}{b} + \frac{2c}{\pi}\left[\sin\frac{\pi a}{b}\cos\frac{\pi x}{b} + \frac{1}{2}\sin\frac{2\pi a}{b}\cos\frac{2\pi x}{b} + \ldots\right]\right]$$

27. Obtain sine series for

$f(x) = mx, \quad 0 < x \le \frac{\pi}{2}$

$= m(\pi - x), \quad \frac{\pi}{2} \le x < \pi$ **(M.U. 2004)**

$$\left[\textbf{Ans. : } f(x) = \frac{4m}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right]\right]$$

28. Obtain half range cosine series for $f(x) = \sin\left(\frac{\pi x}{l}\right)$ in $0 < x < l$.

(M.U. 2000)

$$\left[\textbf{Ans. : } f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{1\cdot 3}\cos\frac{2\pi x}{l} + \frac{1}{3\cdot 5}\cos\frac{4\pi x}{l} + \ldots\right]\right]$$

29. Obtain half-range cosine series for $f(x) = (x-1)^2$ in $0 < x < 1$.

(M.U. 2002)

Hence, find $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$.

[**Ans. :** $f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2}$. Now, put $x = 0$ and then $x = 1$]

30. Obtain half-range sine series for $f(x) = \pi x - x^2$ in $(0, \pi)$ and hence deduce that

(i) $\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots = \frac{\pi^2}{960}$

(ii) $\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{4^6} + \ldots = \frac{\pi^2}{945}$.

$$\left[\textbf{Ans. : } f(x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\right]\right]$$

31. Obtain half-range sine series for $f(x)$ given by

$$f(x) = \begin{cases} kx, & 0 < x < (l/2) \\ k(l-x), & (l/2) < x < l \end{cases}$$

Hence, deduce that $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$.

$$\left[\textbf{Ans. : } f(x) = \frac{4kl}{\pi^2}\left[\frac{\sin(\pi x/l)}{1^2} - \frac{\sin(3\pi x/l)}{3^2} + \frac{\sin(5\pi x/l)}{5^2} + \ldots\right]\right]$$

(B) **1.** If $x^2 = \frac{4l^2}{3} + \frac{4l^2}{\pi^2}\sum\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right) - \frac{4l^2}{\pi}\sum\frac{1}{n}\sin\frac{n\pi x}{l}$

in $0 < x < 2l$, find the sum of the series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

(M.U. 2003) [**Ans. :** $\pi^2/6$]

2. If $\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum\frac{1}{n^2}\cos nx$ in $0 < x < 2\pi$, find the sum of the series $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots$ [**Ans. :** $\frac{\pi^4}{90}$]

3. If $\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\cos\frac{2x}{3} + \cos\frac{4x}{15} + \cos\frac{6x}{35} + \ldots\right)$

in $0 \le x \le \pi$, find the sum of the series $\frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \ldots$

[**Ans.:** $\frac{\pi^2 - 8}{16}$]

4. If $x - x^2 = \frac{8}{\pi^3}\left[\frac{1}{1^3}\sin \pi x + \frac{1}{3^3}\sin 3\pi x + \frac{1}{5^3}\sin 5\pi x + \ldots\right]$

in $0 < x < 1$, find the sum of the series $\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots$ [**Ans.:** $\frac{\pi^6}{945}$]

5. If $x^2 = \frac{\pi}{3} + 4\sum(-1)^n \cdot \frac{\cos nx}{n^2}$ for $-\pi < x < \pi$, prove that

$$\sum \frac{1}{n^4} = \frac{\pi^4}{90}.$$ **(M.U. 2003)**

EXERCISE

Theory

1. Define Fourier Series and evaluate the Fourier Constants. **(M.U. 1999)**
2. State Parseval's identity for a function $f(x)$ in $(-l, l)$. **(M.U. 1998, 2003)**
3. Derive Euler's formulae. **(M.U. 1997)**
4. State and prove Parseval's identity for the function $f(x)$ in the interval $(c, c + 2\pi)$. **(M.U. 1993)**
5. Define Fourier Series and State Dirichlet's Conditions.
6. State and prove Parseval's identity for half range cosine series for $f(x)$ in the interval $(0, l)$.
7. State Parseval's identity for a function $f(x)$ in $(-\pi, \pi)$. **(M.U. 2000)**

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4

Complex Form of Fourier Series

1. Introduction

In this chapter we shall first derive complex from of Fourier Series. Then we shall define Fourier Integrals and solve some simple problems based on this.

2. Complex Form Of Fourier Series

Let $f(x)$ be defined in the interval $(C, C + 2l)$. The complex form of Fourier Series for $f(x)$ in this interval is given by

$$\boxed{\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \\ C_n &= \frac{1}{2l}\int_C^{C+2l} f(x)\, e^{-in\pi x/l} dx \end{aligned}}$$

where, $\quad n = 0, \pm 1, \pm 2, \ldots$

Proof : Consider

$$f(x) = a_0 + \sum_1^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_1^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, a_0, a_n and b_n are as given in (3) on page 7-51.

$$= a_0 + \sum_1^{\infty} a_n\left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + \sum_1^{\infty} b_n\left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)$$

$$= a_0 + \sum_1^{\infty}\left(\frac{a_n - ib_n}{2}\right) e^{in\pi x/l} + \sum_1^{\infty}\left(\frac{a_n + ib_n}{2}\right) e^{-in\pi x/l}$$

$$= C_0 + \sum_1^{\infty} C_n e^{in\pi x/l} + \sum_1^{\infty} C_{-n} e^{-in\pi x/l}$$

where, $C_n = \frac{a_n - ib_n}{2}$, $C_{-n} = \frac{a_n + ib_n}{2}$

$$\therefore \quad f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}$$

where, $C_0 = a_0 = \dfrac{1}{2l}\displaystyle\int_C^{C+2l} f(x)\,dx$

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{2l}\left[\int_C^{C+2l} f(x)\cos\frac{n\pi x}{l}\,dx - i\int_C^{C+2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx \quad \left[\because e^{-i\theta} = \cos\theta - i\sin\theta\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\,e^{-in\pi x/l}dx \qquad \text{.......... (1)}$$

$$C_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2l}\left[\int_C^{C+2l} f(x)\cos\frac{n\pi x}{l}\,dx + i\int_C^{C+2l} f(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)dx \quad \left[\because e^{i\theta} = \cos\theta + i\sin\theta\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\,e^{in\pi x/l}dx \qquad \text{.......... (2)}$$

Combining these results (1) and (2), we get

$$C_n = \frac{1}{2l}\int_C^{C+2l} f(x)\,e^{-in\pi x/l}dx \qquad n = 0, \pm 1, \pm 2, \ldots.$$

Cor. 1 : If the interval is C to $C + 2\pi$ replacing l by π in the above result

$$\boxed{\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} C_n e^{inx} \\ \text{where, } C_n &= \frac{1}{2\pi}\int_C^{C+2\pi} f(x)\,e^{-inx}dx \end{aligned}}$$

Cor. 2 : If the interval is $(0, 2l)$, putting $C = 0$ in the above result

$$\boxed{\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \\ \text{where, } C_n &= \frac{1}{2l}\int_0^{2l} f(x)\,e^{-in\pi x/l}dx \end{aligned}}$$

Cor. 3 : If the interval is $(0, 2\pi)$, putting $l = \pi$ in the above corollary 2,

$$\boxed{\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} C_n e^{inx} \\ \text{where, } C_n &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,e^{-inx}dx \end{aligned}}$$

Cor. 4 : If the interval is $(-l, l)$, putting $C = -l$ in the above result

$$\boxed{\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \\ \text{where, } C_n &= \frac{1}{2l}\int_{-l}^{l} f(x)\,e^{-in\pi x/l}dx \end{aligned}}$$

Cor. 5 : If the interval is $(-\pi, \pi)$, putting $l = \pi$ in corollary 4,

$$\boxed{\begin{aligned} f(x) &= \sum_{-\infty}^{\infty} C_n e^{inx} \\ \text{where, } C_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}dx \end{aligned}}$$

Ex. 1 : Obtain complex form of Fourier Series for $f(x) = e^{ax}$ in $(-\pi, \pi)$ where a is not an integer. **(B.U. 1993, 94, 98, 2001, 02, 04, 05)**

Hence deduce that when α is a constant other than an integer

(i) $\cos\alpha x = \dfrac{\sin\pi\alpha}{\pi}\displaystyle\sum \frac{(-1)^n\,\alpha}{(\alpha^2 - n^2)}\cdot e^{inx}$

(ii) $\sin\alpha x = \dfrac{\sin\pi\alpha}{i\pi}\displaystyle\sum \frac{(-1)^n\cdot n}{(\alpha^2 - n^2)}\cdot e^{inx}$ **(M.U. 2001, 02)**

Sol. : By corollary 5 above, the complex form of $f(x) = e^{ax}$ is given by

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$$

where, $C_n = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,e^{-inx}dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\cdot e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a - in}\right]_{-\pi}^{\pi} = \frac{1}{2\pi(a - in)}\left[e^{(a-in)\pi} - e^{-(a-in)\pi}\right]$$

$$= \frac{1}{2\pi(a - in)}\left[e^{a\pi}\cdot e^{-in\pi} - e^{-a\pi}\cdot e^{in\pi}\right]$$

But $\quad e^{\pm in\pi} = \cos(\pm n\pi) + i\sin(\pm n\pi)$

$$= (-1)^n + i(0) = (-1)^n$$

$$\therefore \quad C_n = \frac{1}{2\pi(a - in)}\left[(-1)^n e^{a\pi} - (-1)^n e^{-a\pi}\right]$$

$$= \frac{(-1)^n}{\pi(a-in)}\left(\frac{e^{a\pi}-e^{-a\pi}}{2}\right) = \frac{(-1)^n}{\pi(a-in)} \sin h\, a\pi$$

$$= \frac{(-1)^n \sin h\, a\pi}{\pi(a-in)} \cdot \frac{(a+in)}{(a+in)} = \frac{(-1)^n \sin h\, a\pi\,(a+in)}{\pi(a^2+n^2)}$$

Hence, $$e^{ax} = \sum_{-\infty}^{\infty} \frac{(-1)^n \sin h\, a\pi \cdot (a+in)}{\pi(a^2+n^2)} e^{inx} \qquad \text{.......... (i)}$$

For deductions, replace a by $i\alpha$ in (i)

$$\therefore \quad e^{i\alpha x} = \sum \frac{(-1)^n \sin h\, i\alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (i\alpha+in)\cdot e^{inx}$$

$$= \sum \frac{(-1)^n (i) \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (i\alpha+in)\cdot e^{inx}$$

$$= \sum \frac{(-1)^n \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-\alpha-n)\cdot e^{inx}$$

Now, replace a by $-i\alpha$ in (i)

$$\therefore \quad e^{-i\alpha x} = \sum \frac{(-1)^n \sin h(-i\alpha\pi)}{\pi(-\alpha^2+n^2)} \cdot (-i\alpha+in)\cdot e^{inx}$$

$$= \sum \frac{(-1)^n (-i) \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-i\alpha+in)\cdot e^{inx}$$

$$= \sum \frac{(-1)^n \sin \alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-\alpha+n)\cdot e^{inx}$$

$$\therefore \quad \cos \alpha x = \frac{e^{i\alpha x}+e^{-i\alpha x}}{2} = \sum \frac{(-1)^n \sin\alpha\pi}{\pi(-\alpha^2+n^2)} \cdot (-\alpha)\cdot e^{inx}$$

$$= \frac{\sin\alpha\pi}{\pi} \sum \frac{(-1)^n \cdot \alpha}{(\alpha^2-n^2)} \cdot e^{inx}$$

$$\therefore \quad \sin \alpha x = \frac{e^{i\alpha x}-e^{-i\alpha x}}{2\pi i} = \sum \frac{(-1)^n \sin\alpha\pi}{\pi i(-\alpha^2+n^2)} \cdot (-n)\cdot e^{inx}$$

$$= \frac{\sin\alpha\pi}{\pi i} \sum \frac{(-1)^n \cdot n}{(\alpha^2-n^2)} \cdot e^{inx}.$$

Ex. 2 : Obtain the complex form of Fourier Series for $f(x) = e^{-ax}$ in $(-\pi, \pi)$.

Sol. : Changing the sign of a, we get from the above result

$$f(x) = \sum_{-\infty}^{\infty} \frac{(-1)^n \sin h(-a\pi)\cdot(-a+in)}{\pi(a^2+n^2)} e^{inx}$$

But $$\sin h(-a\pi) = \frac{e^{-a\pi}-e^{a\pi}}{2} = -\frac{e^{a\pi}-e^{-a\pi}}{2} = -\sin h\, a\pi$$

$$\therefore \quad e^{-ax} = \sum \frac{(-1)^n \sin h\, a\pi\cdot(a-in)}{\pi(a^2+n^2)} e^{inx}$$

Or proceeding as above obtain the result independently.

Ex. 3 : Obtain the complex form of Fourier Series for $f(x) = \cos h\, ax$ in $(-\pi, \pi)$ where a is not an integer.

Sol. : As proved in example 1,

$$e^{ax} = \sum \frac{(-1)^n \sin h\, a\pi\cdot(a+in)}{\pi(a^2+n^2)} e^{inx}$$

Changing the sign of a, we get,

$$e^{-ax} = \sum \frac{(-1)^n \sin h(-a\pi)\cdot(-a+in)}{\pi(a^2+n^2)} e^{inx}$$

But $\sin h(-a\cdot\pi) = -\sin h\, a\pi$.

$$e^{-ax} = \sum \frac{(-1)^n \sin h\, a\pi\cdot(a-in)}{\pi(a^2+n^2)} e^{inx}$$

Now, $$\cos h\, ax = \frac{e^{ax}+e^{-ax}}{2}$$

$$= \frac{\sin h\, a\pi}{2}\left[\sum \frac{(-1)^n(a+in)}{(a^2+n^2)} e^{inx} + \sum \frac{(-1)^n(a-in)}{(a^2+n^2)} e^{inx}\right]$$

$$\cos h\, ax = a\cdot \sin h\, a\pi \sum \frac{(-1)^n}{(a^2+n^2)} e^{in\pi x}$$

Ex. 4 : Obtain the complex form of Fourier Series for $f(x) = e^{ax}$ in $(-l, l)$.

(B.U. 1993, 2003)

Sol. : By corollary 4 above, the complex form of $f(x) = e^{ax}$ is given by

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \qquad \text{.......... (1)}$$

where, $$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-in\pi x/l}$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{ax}\cdot e^{-in\pi x/l}\cdot dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{(a-in\pi/l)x}\, dx = \frac{1}{2l}\left[\frac{e^{(a-in\pi/l)x}}{(a-in\pi/l)}\right]_{-l}^{l}$$

$$= \frac{1}{2l}\left[\frac{e^{(a-in\pi/l)l} - e^{-(a-in\pi/l)l}}{(a-in\pi/l)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{al}\cdot e^{-in\pi} - e^{-al}\cdot e^{in\pi}}{(al-in\pi)}\right]$$

Now, $e^{\pm in\pi} = \cos(\pm n\pi) + i\sin(\pm n\pi)$

$$= (-1)^n + i - 0 = (-1)^n$$

$$\therefore \quad C_n = \frac{e^{al}(-1)^n - e^{-al}(-1)^n}{2(al-in\pi)} = \frac{(-1)^n(e^{al}-e^{-al})}{2(al-in\pi)}$$

$$= \frac{(-1)^n\cdot\sin h\,al}{al-in\pi}\cdot\frac{al+in\pi}{al+in\pi}$$

$$= \frac{(-1)^n\sin h\,al\,(al+in\pi)}{a^2l^2+n^2\pi^2}$$

$$\therefore \quad e^{ax} = \sum_{-\infty}^{\infty}\frac{(-1)^n\sin h\,al\cdot(al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

Ex. 5 : Obtain the complex form of Fourier Series for $f(x) = \cos h\,ax$ in $(-l, l)$. **(M.U. 2002, 05)**

Sol. : By example (4) above

$$e^{ax} = \sum\frac{(-1)^n\sin h\,al\cdot(al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

Changing the sign of a, we get

$$e^{-ax} = \sum\frac{(-1)^n\sin h(-al)(-al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

But $\sin h(-x) = \dfrac{e^{-x}-e^{x}}{2} = -\left(\dfrac{e^{x}-e^{-x}}{2}\right) = -\sin h\,x$

$$\therefore \quad e^{-ax} = \sum\frac{(-1)^n\sin h\,al\,(al-in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

$$\therefore \quad \cos h\,ax = \frac{e^{ax}+e^{-ax}}{2}$$

$$= \frac{\sin h\,al}{2}\left[\sum\frac{(-1)^n(al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l} + \sum\frac{(-1)^n(al-in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}\right]$$

$$= al\sin h\,al\sum\frac{(-1)^n}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

Ex. 6 : Obtain complex form of Fourier Series for $f(x) = \sin h\,ax$ in $(-l, l)$.

Sol. : Using the results obtained in example 5, we get

$$\sin h\,ax = \frac{e^{ax}-e^{-ax}}{2}$$

$$= \frac{\sin h\,al}{2}\left[\sum\frac{(-1)^n(al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l} - \sum\frac{(-1)^n(al-in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}\right]$$

$$= \sin h\,al\sum\frac{(-1)^n in\pi}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

Ex. 7 : Obtain complex form of Fourier Series for $f(x) = \cos h\,ax + \sin h\,ax$ in $(-l, l)$. **(M.U. 2002)**

Sol. : By using the results obtained in examples 5 and 6,

$$f(x) = \cos h\,ax + \sin h\,ax$$

$$= \sin h\,al\sum\frac{(-1)^n(al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

Ex. 8 : Find complex form of Fourier Series for $\cos ax$, where a is not an integer in $(-\pi, \pi)$.

Sol. : We have $\cos ax = \dfrac{e^{aix}+e^{-aix}}{2}$

Replacing a by ai in example 1, we get

$$e^{aix} = \sum_{-\infty}^{\infty}\frac{(-1)^n\sin h\,ai\pi\cdot(ai+in)}{\pi(-a^2+n^2)}e^{inx} \quad \left[\because -a^2+n^2 \neq 0\right]$$

$$= \frac{-i\sin h\,ai\pi}{\pi}\sum\frac{(-1)^n(a+n)}{(a^2-n^2)}e^{inx}$$

Since, $\sin h\,ix = i\sin x$, we get

$$e^{aix} = \frac{\sin a\pi}{\pi}\sum\frac{(-1)^n(a+n)}{(a^2-n^2)}e^{inx}$$

Changing the sign of a,

$$e^{-aix} = \frac{\sin(-a\pi)}{\pi}\sum\frac{(-1)^n(-a+n)}{(a^2-n^2)}e^{inx}$$

$$= -\frac{\sin a\pi}{\pi}\sum\frac{(-1)^n(-a+n)}{(a^2-n^2)}e^{inx}$$

$$= \frac{\sin a\pi}{\pi} \sum \frac{(-1)^n (a-n)}{(a^2-n^2)} e^{inx}$$

$$\therefore \quad \cos ax = \frac{e^{aix} + e^{-aix}}{2}$$

$$= \frac{\sin a\pi}{2\pi}\left[\sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx} + \sum \frac{(-1)^n (a-n)}{(a^2-n^2)} e^{inx}\right]$$

$$= \frac{\sin a\pi}{2\pi} \sum \frac{(-1)^n \cdot 2a}{(a^2-n^2)} e^{inx}$$

$$= \frac{a \sin a\pi}{\pi} \sum \frac{(-1)^n}{(a^2-n^2)} e^{inx}$$

Ex. 9 : Find complex from of Fourier Series for $\sin ax$, where a is not an integer in $(-\pi, \pi)$. **(B.U. 2004)**

Sol. : Using the results obtained in the above example, we get,

$$\sin ax = \frac{e^{aix} - e^{-aix}}{2i}$$

$$= \frac{\sin a\pi}{2\pi i}\left[\sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx} - \sum \frac{(-1)^n (a-n)}{(a^2-n^2)} e^{inx}\right]$$

$$= \frac{\sin a\pi}{\pi i} \sum (-1)^n \cdot \frac{n}{(a^2-n^2)} \cdot e^{inx}$$

Ex. 10 : Obtain the complex form of Fourier series for $f(x) = e^{ax}$ in $(0, a)$. **(M.U. 2003)**

Sol. : By corollary (2), page 2-2, putting $2l = a$ i.e. $l = a/2$, we get

$$f(x) = \sum_{-\infty}^{\infty} C_n \cdot e^{2in\pi x/a}$$

$$\text{where} \quad C_n = \frac{1}{a}\int_0^a e^{ax} \cdot e^{-2in\pi x/a}\, dx$$

$$\therefore \quad C_n = \frac{1}{a}\int_0^a e^{(a-2in\pi/a)x} dx$$

$$= \frac{1}{a}\left[\frac{e^{(a-2in\pi/a)x}}{(a-2in\pi/a)}\right]_0^a$$

$$= \frac{1}{a} \cdot \frac{a}{(a^2-2in\pi)} \cdot \left[e^{(a^2-2in\pi)} - 1\right]$$

$$= \frac{1}{(a^2-2in\pi)}\left[e^{a^2} \cdot e^{-2in\pi} - 1\right]$$

$$= \frac{1}{(a^2-2in\pi)}(e^{a^2} - 1) \quad \left[\because e^{-2in\pi} = \cos 2n\pi - i\sin 2n\pi = 1\right]$$

$$\therefore \quad e^{ax} = (e^{a^2} - 1)\sum_{-\infty}^{\infty} \frac{e^{2in\pi x/a}}{(a^2-2in\pi)}.$$

Ex. 11 : Find the complex form of Fourier Series for

$$f(x) = \begin{cases} 0, & 0 < x < l \\ a, & l < x < 2l \end{cases}$$

Sol. : By corollary 2, the complex form of Fourier Series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{in\pi x/l}$$

$$\text{where, } C_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-in\pi x/l}\, dx$$

$$\therefore \quad C_n = \frac{1}{2l}\left[\int_0^l 0 \cdot dx + \int_l^{2l} a\, e^{-in\pi x/l}\, dx\right] \qquad \text{.......... (1)}$$

$$= \frac{a}{2l}\int_l^{2l} e^{-in\pi x/l}\, dx = \frac{a}{2l}\left[\frac{e^{-in\pi x/l}}{-in\pi/l}\right]_l^{2l}$$

$$= \frac{-a}{2in\pi}\left[e^{-2in\pi} - e^{-in\pi}\right] \quad \text{except when } n = 0. \qquad \text{.......... (2)}$$

Case 1 : Where $n = 0$ from (1)

$$C_0 = \frac{1}{2l}\left[\int_0^l 0\, dx + \int_l^{2l} a\, e^0\, dx\right]$$

$$\therefore \quad C_0 = \frac{1}{2l}\int_l^{2l} a\, dx = \frac{a}{2l}[2l - l]$$

$$= \frac{al}{2l} = \frac{a}{2}$$

Case 2 : When $n = \pm 1, \pm 3, \ldots\ldots$ from (2)

$$C_1 = \frac{-a}{2i\pi}\left[e^{-2i\pi} - e^{-i\pi}\right]$$

$$= \frac{-a}{2i\pi}[\cos(-2\pi) + i\sin(-2\pi) - \cos(-\pi) - i\sin(-\pi)]$$

$$= \frac{-a}{2i\pi}[1 + i(0) - (-1) + i(0)] = \frac{-a}{2i\pi} \cdot 2 = \frac{ai}{\pi}$$

$$C_{-1} = \frac{a}{2i\pi}[\cos 2\pi + i\sin 2\pi - \cos\pi - i\sin\pi]$$

$$= \frac{a}{2i\pi}[1 + i(0) - (1) + i\,0] = \frac{a}{2i\pi}\cdot 2 = -\frac{ai}{\pi}$$

Similarly, $C_3 = \frac{ia}{3\pi}$, $C_{-3} = -\frac{ai}{3\pi}$

$C_5 = \frac{ia}{5\pi}$, $C_{-5} = -\frac{ai}{5\pi}$

Case 3 : When $n = \pm 2, \pm 4, \ldots$

$$C_2 = \frac{-a}{4i\pi}\left[e^{-4i\pi} - e^{-2i\pi}\right]$$

$$= \frac{-a}{4i\pi}[\cos(-4\pi) + i\sin(-4\pi) - \cos(-2\pi) + i\sin(-2\pi)]$$

$$= \frac{-a}{4i\pi}[1 + i(0) - (1) - i(0)] = 0$$

Similarly, $C_{-2} = 0$ and $C_4 = C_{-4} = C_6 = C_{-6} = \ldots$

$$\therefore \quad f(x) = \frac{a}{2} + \frac{ai}{\pi}\left[(e^u - e^{-u}) + \frac{1}{3}(e^{3u} - e^{-3u}) + \ldots\right] \text{ where } u = \frac{i\pi x}{l}$$

Ex. 12 : If $f(x) = 1 \quad 0 < x < 1$

$= 0 \quad 1 < x < 2$

$f(x + 2) = f(x)$, find complex from of Fourier Series. **(M.U. 1996)**

Sol. : By corollary 2, the complex form of Fourier Series in $(0, 2l)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$$

where, $C_n = \frac{1}{2l}\int_0^{2l} f(x)\, e^{-in\pi x/l}\, dx$

Since, in this case $l = 1$,

$$\therefore \quad C_n = \frac{1}{2}\int_0^2 f(x)\, e^{-in\pi x} dx \qquad \ldots\ldots\ldots (1)$$

$$= \frac{1}{2}\left[\int_0^1 1\cdot e^{-in\pi x}dx + \int_1^2 0\cdot e^{-in\pi x}dx\right]$$

$$= \frac{1}{2}\left[\frac{e^{-in\pi x}}{-in\pi}\right]_0^1 = \frac{1}{-2in\pi}\left[e^{-in\pi} - 1\right]$$

$$= \frac{1}{2in\pi}\left[1 - e^{-in\pi}\right] \text{ except when } n = 0. \qquad \ldots\ldots\ldots (2)$$

Case 1 : When $n = 0$, from (1)

$$C_0 = \frac{1}{2}\left[\int_0^1 1\cdot dx + \int_1^2 0\cdot dx\right]$$

$$= \frac{1}{2}[x]_0^1 = \frac{1}{2}$$

Case 2 : When $n = \pm 1, \pm 3, \pm 5, \ldots$ from (2)

$$C_1 = \frac{1}{2i\pi}\left[1 - e^{-i\pi}\right] = \frac{1}{2i\pi}[1 - \{\cos\pi - i\sin\pi\}]$$

$$= \frac{1}{2i\pi}[1 - (-1)] = \frac{2}{2i\pi} = \frac{1}{i\pi}$$

$$C_{-1} = \frac{1}{-2i\pi}\left[1 - e^{i\pi}\right] = -\frac{1}{2i\pi}[1 - \{\cos\pi + i\sin\pi\}]$$

$$= -\frac{1}{2i\pi}[1 - (-1)] = -\frac{1}{i\pi}$$

Similarly, $C_3 = \frac{1}{3i\pi}$, $C_{-3} = -\frac{1}{3i\pi}$,

$C_5 = \frac{1}{5i\pi}$, $C_{-5} = -\frac{1}{5i\pi}, \ldots$

Case 3 : When $n = \pm 2, \pm 4, \ldots$ from (2)

$$C_2 = \frac{1}{4i\pi}\left[1 - e^{-2i\pi}\right] = \frac{1}{4i\pi}[1 - \{\cos 2\pi + i\sin 2\pi\}]$$

$$= \frac{1}{4i\pi}[1 - 1] = 0$$

Similarly, $C_{-2} = 0$ and $C_4 = C_{-4} = C_6 = C_{-6} = \ldots = 0$

Hence, $f(x) = \frac{1}{2} + \frac{1}{i\pi}(e^u - e^{-u}) + \frac{1}{3i\pi}(e^{3u} - e^{-3u}) + \ldots$

where, $u = i\pi x$

$$= \frac{1}{2} + \frac{2}{i\pi}\left[\left(\frac{e^u - e^{-u}}{2}\right) + \frac{1}{3}\frac{(e^{3u} - e^{-3u})}{2} + \ldots\right]$$

$$= \frac{1}{2} + \frac{2}{i\pi}\left[\frac{\sin hu}{1} + \frac{\sin h\,3u}{3} + \ldots\right] + \ldots \qquad \text{where, } u = i\pi x$$

EXERCISE

1. Find complex form of Fourier Series for $f(x) = e^{-x}$ in $(-1, 1)$.

$$\left[\textbf{Ans. :}\ f(x) = \sum_{-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)\sin h1}{1 + n^2\pi^2}\, e^{in\pi x}\right]$$

2. Find the complex form of $f(x) = e^x$ in $(-\pi, \pi)$. **(B.U. 1999, 2000, 04)**

$$\left[\textbf{Ans. : } f(x) = \sum \frac{(-1)^n \sinh \pi (1+in)}{\pi(1^2+n^2)} e^{inx}\right]$$

3. Obtain complex form of Fourier series for $f(x) = e^{ax}$, $-1 < x < 1$. **(B.U. 1996, 2003)**

$$\left[\textbf{Ans. : } f(x) = \sum \frac{(-1)^n \sinh a(a+in\pi)}{a^2+n^2\pi^2} e^{in\pi}\right]$$

4. Find the complex form of Fourier Series for $f(x) = \sin hx$ in $(-l, l)$. **(B.U. 1997)**

$$\left[\textbf{Ans. : } f(x) = \sinh l \cdot (i\pi) \sum \frac{(-1)^n n}{l^2+n^2\pi^2} e^{in\pi x/l}\right]$$

5. Find the complex form of Fourier Series for $f(x) = \cos h\, 2x + \sin h\, 2x$ in $(-5, 5)$. **(B.U. 1999, 2003)**

(**Hint :** In solved example 6, put $a = 2$ and $l = 5$.)

6. Find the complex form of Fourier Series of $\cos h\, 3x + \sin h\, 3x$ in $(-\pi, \pi)$. **(B.U. 1999, 2003)**

(**Hint :** In solved example 7, put $a = 3$, $l = \pi$.)

7. Obtain the complex form of Fourier Series for $f(x) = e^{2x}$ in $(0, 2)$.

(**Hint :** Put $a = 2$ in solved example 10.) **(M.U. 2003)**

3. Orthogonality, Orthonormality

We first define these terms and then solve some problems based on this definition.

Definition 1 : A set of functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x) \ldots$ is said to be **orthogonal** on (a, b) if

$$\int_a^b f_m(x)\, f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$$

In other words a set of functions $f_1(x), f_2(x), \ldots f_n(x) \ldots$ is orthogonal on (a, b) if

$$\boxed{\begin{aligned} &\int_a^b f_m(x)\, f_n(x)\, dx = 0 \quad \text{if } m \neq n \\ \text{and} \quad &\int_a^b [f_m(x)]^2 dx \neq 0 \end{aligned}}$$

Definition 2 : A set of functions $f_1(x), f_2(x), f_3(x), \ldots, f_n(x) \ldots$ is said to be **orthonormal** on (a, b) if

$$\int_a^b f_m(x)\, f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ = 1, & \text{if } m = n \end{cases}$$

In other words a set of functions $f_1(x), f_2(x), f_3(x), \ldots f_n(x) \ldots$ is said to be **orthonormal** on (a, b) if

$$\boxed{\begin{aligned} &\int_a^b f_m(x)\, f_n(x)\, dx = 0 \quad \text{if } m \neq n \\ \text{and} \quad &\int_a^b [f_m(x)]^2 dx = 1 \end{aligned}}$$

Note : Every orthonormal set of functions is orthogonal but the converse may not be true.

Ex. 1 : Show that the set of functions $\cos nx$, $n = 1, 2, 3, \ldots$ is orthogonal on $(0, 2\pi)$. **(B.U. 1994)**

Sol. : We have $f_n(x) = \cos nx$

$$\therefore \int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx = \int_0^{2\pi} \cos mx \cdot \cos nx\, dx$$

$$= \frac{1}{2} \int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x]\, dx$$

$$= \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{2\pi}$$

Now, two cases arise

Case 1 : When $m \neq n$, then

$$\int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx = 0$$

Case 2 : When $m = n$, then

$$\int_0^{2\pi} f_n(x) \cdot f_n(x)\, dx = \int_0^{2\pi} \cos^2 nx\, dx$$

$$\therefore \int_0^{2\pi} [f_n(x)]^2 dx = \int_0^{2\pi} \left(\frac{1+\cos 2nx}{2}\right) dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_0^{2\pi} = \pi \neq 0 \qquad \ldots\ldots\ldots (1)$$

Since, $\int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$

the given set of functions are orthogonal over $[0, 2\pi]$.

Ex. 2 : How can you construct orthonormal set of functions from the set given in the above example number 1?

Sol. : If the set of functions is to be orthonormal, we should have

$$\int_0^{2\pi} [f_n(x)]^2 dx = 1$$

For this we divide (1) by π and write it as

$$\int_0^{2\pi} \frac{1}{\pi}[f_n(x)]^2\,dx = \pi \cdot \frac{1}{\pi} = 1$$

i.e. $$\int_0^{2\pi} \frac{1}{\sqrt{\pi}} f_n(x) \cdot \frac{1}{\sqrt{\pi}} f_n(x)\,dx = 1$$

This is obviously an orthonormal set, where

$$\phi_n(x) = \frac{1}{\sqrt{\pi}} \cos nx$$

Hence, the required orthonormal set is

$$\frac{1}{\sqrt{\pi}}\cos x,\ \frac{1}{\sqrt{\pi}}\cos 2x,\ \frac{1}{\sqrt{\pi}}\cos 3x,\ \ldots..$$

Ex. 3 : Show that the set of functions $\sin(2n+1)x$, $n = 0, 1, 2, \ldots..$ is orthogonal over $[0, \pi/2]$. Hence, construct orthonormal set of functions.

(B.U. 1993, 2002, 04)

Sol. : We have $f(x) = \sin(2n+1)x$.

$$\therefore \int_0^{\pi/2} f_m(n) \cdot f_n(x)\,dx = \int_0^{\pi/2} \sin(2m+1)x \cdot \sin(2n+1)x\,dx$$

$$= -\frac{1}{2}\int_0^{\pi/2} [\cos(2m+2n+2)x - \cos(2m-2n)x]\,dx$$

$$= -\frac{1}{2}\left[\frac{\sin(2m+2n+2)x}{(2m+2n+2)} - \frac{\sin(2m-2n)x}{(2m-2n)}\right]_0^{\pi/2}$$

Now, two cases arise

Case 1 : If $m \neq n$, then

$$\int_0^{\pi/2} f_m(x) \cdot f_n(x) = 0$$

Case 2 : If $m = n$, then

$$\int_0^{\pi/2} f_n(x) \cdot f_n(x)\,dx = \int_0^{\pi/2} \sin^2(2n+1)x\,dx$$

$$\therefore \int_0^{\pi/2} [f_n(x)]^2\,dx = \int_0^{\pi/2} \left(\frac{1-\cos 2(2n+1)x}{2}\right)dx$$

$$= \left[\frac{x}{2} - \frac{\sin 2(2n+1)x}{2(2n+1)}\right]_0^{\pi/2} = \frac{\pi}{4} \neq 0 \qquad \ldots\ldots\ldots. (1)$$

Since,

$$\int_0^{\pi/2} f_m(x) \cdot f_n(x)\,dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$$

the given set of functions is orthogonal over $[0, \pi/2]$.

Now, if the set is to be orthonormal, then we should have,

$$\int_0^{\pi/2} [f_n(n)]^2\,dx = 1$$

For this, we divide (1) by $\pi/4$ and write it as

$$\int_0^{\pi/2} \frac{4}{\pi}[f_n(x)]^2\,dx = \frac{4}{\pi} \cdot \frac{\pi}{4} = 1$$

i.e. $$\int_0^{\pi/2} \frac{2}{\sqrt{\pi}} f_n(x) \cdot \frac{2}{\sqrt{\pi}} f_n(x)\,dx = 1$$

This is obviously an orthonormal set, where

$$\phi_n(x) = \frac{2}{\sqrt{\pi}} \sin(2n+1)x$$

Hence, the required orthonormal set of functions is

$$\frac{2}{\sqrt{\pi}}\sin x,\ \frac{2}{\sqrt{\pi}}\sin 3x,\ \frac{2}{\sqrt{\pi}}\sin 5x,\ \ldots..$$

Ex. 4 : Is $S = \left\{\sin\left(\frac{\pi x}{4}\right), \sin\left(\frac{3\pi x}{4}\right), \sin\left(\frac{5\pi x}{4}\right), \ldots..\right\}$ orthogonal in (0, 1) ?

(M.U. 2004)

Sol. : We have

$$\int_0^1 f_m(x) \cdot f_n(x)\,dx = \int_0^1 \sin\frac{(2m+1)\pi x}{4} \sin\frac{(2n+1)\pi x}{4} \cdot dx$$

$$= -\frac{1}{2}\left[\int_0^1\left[\cos\frac{(m+n+1)\pi x}{2} - \cos\frac{(m-n)\pi x}{2}\right]dx\right]$$

$$= -\frac{1}{2}\left[\sin\frac{(m+n+1)\pi x}{2} \cdot \frac{2}{(m+n+1)\pi}\right.$$

$$\left. - \sin\frac{(m-n)\pi x}{2} \cdot \frac{2}{(m-n)\pi}\right]_0^1 \neq 0$$

$\therefore$ S is not orthogonal.

Ex. 5 : Show that the set of functions

$$\sin\left(\frac{\pi x}{2L}\right),\ \sin\left(\frac{3\pi x}{2L}\right),\ \sin\left(\frac{5\pi x}{2L}\right),\ \ldots.. \text{ is orthogonal over } (0, L).$$

(B.U. 1996, 2002, 05)

Sol. : We have $f_n(x) = \sin\frac{(2n+1)\pi x}{2L}$, $\quad n = 0, 1, 2, \ldots.$

$$\therefore \int_0^L f_m(x) \cdot f_n(x)\,dx = \int_0^L \sin\frac{(2m+1)\pi x}{2L} \cdot \sin\frac{(2n+1)\pi x}{2L}\,dx$$

$$= -\frac{1}{2}\int_0^L \left[\cos\left(\frac{2m+2n+2}{2L}\right)\pi x - \cos\left(\frac{2m-2n}{2L}\right)\pi x\right] dx$$

$$= -\frac{1}{2}\int_0^L \left[\cos\left(\frac{m+n+1}{L}\right)\pi x - \cos\left(\frac{m-n}{L}\right) x\right] dx$$

$$= -\frac{1}{2}\left[\frac{\sin\{(m+n+1)/L\}\,\pi x}{(m+n+1)\,\pi/L} - \frac{\sin\{(m-n)/L\}\,\pi x}{(m-n)\,\pi/L}\right]_0^L$$

$$= -\frac{1}{2}\left[\frac{\sin(m+n+1)\,\pi}{(m+n+1)\,\pi/L} - \frac{\sin(m-n)\,\pi}{(m-n)\,\pi/L}\right] \qquad \text{.......... (1)}$$

Now, two cases arise

Case 1 : If $m \neq n$, then since m, n are integers from (1)

$$\int_0^L f_m(x) \cdot f_n(x)\, dx = 0$$

Case 2 : If $m = n$, then

$$\int_0^L f_n(x) \cdot f_n(x)\, dx = \int_0^L \sin^2 \frac{(2n+1)\,\pi x}{2L}\, dx$$

$$\therefore \quad \int_0^L [f_n(x)]^2\, dx = \int_0^L \left[\frac{1-\cos 2\{(2n+1)/2L\}\,\pi x}{2}\right] dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2\{(2n+1)/2L\}\,\pi x}{(2n+1)\,\pi/L}\right]_0^L$$

$$= \frac{1}{2}\left[x - \frac{\sin\{(2n+1)/L\}\,\pi x}{(2n+1)\,\pi/L}\right]_0^L$$

$$= \frac{L}{2} \neq 0$$

Since, $\int_0^L f_m(x) \cdot f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$

the given set of functions is orthogonal over $[0, L]$.

Ex. 6 : Show that the set of functions

$$1,\ \sin\frac{\pi x}{L},\ \cos\frac{\pi x}{L},\ \sin\frac{2\pi x}{L},\ \cos\frac{2\pi x}{L},\ \ldots..$$

form an orthogonal set in $(-L, L)$ and construct an orthonormal set.

(B.U. 1998, 2002)

Sol. : Let $f_n(x) = \sin\frac{n\pi x}{L}, \quad n = 0, 1, 2, \ldots..$

$$g_n(x) = \cos\frac{n\pi x}{L}, \quad n = 0, 1, 2, \ldots..$$

(a) $$\int_{-L}^L f_m(x) \cdot f_n(x)\, dx = \int_{-L}^L \sin\frac{m\pi x}{L} \cdot \sin\frac{n\pi x}{L}\, dx \qquad \text{.......... (1)}$$

$$= -\frac{1}{2}\int_{-L}^L \left[\cos\frac{(m+n)\,\pi x}{L} - \frac{\cos(m-n)\,\pi x}{L}\right] dx$$

$$= -\frac{1}{2}\left[\frac{\sin\{(m+n)/L\}\,\pi x}{(m+n)\,\pi/L} - \frac{\sin\{(m-n)/L\}\,\pi x}{(m-n)\,\pi/L}\right]_{-L}^L$$

Now, two cases arise

Case 1 : If $m \neq n$, then

$$\int_{-L}^L f_m(x) \cdot f_n(x)\, dx = 0$$

Case 2 : If $m = n$, then from (1)

$$\int_{-L}^L f_n(x)\, dx \cdot f_n(x)\, dx = \int_{-L}^L \sin^2\frac{n\pi x}{L}\, dx$$

$$\therefore \quad \int_{-L}^L [f_n(x)]^2\, dx = \int_{-L}^L \left(\frac{1-\cos(2n\pi x/L)}{2}\right) dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2n\pi x)/L}{2n\pi/L}\right]_{-L}^L = L \neq 0$$

(b) $$\int_{-L}^L g_m(x) \cdot g_n(x)\, dx = \int_{-L}^L \cos\frac{m\pi x}{L} \cdot \cos\frac{n\pi x}{L}\, dx$$

$$= \frac{1}{2}\int_{-L}^L \left[\cos\frac{(m+n)\,\pi x}{L} + \cos\frac{(m-n)\,\pi x}{L}\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin\{(m+n)/L\}\,\pi x}{(m+n)\pi/L} + \frac{\sin\{(m-n)/L\}\,\pi x\}}{(m-n)\,\pi/L}\right]_{-L}^L$$

Again two cases arise

Case 1 : When $m \neq n$, then

$$\int_{-L}^L g_m(x) \cdot g_n(x)\, dx = 0$$

Case 2 : If $m = n$, then

$$\int_{-L}^L g_n(x) \cdot g_n(x)\, dx = \int_{-L}^L \cos^2\frac{n\pi x}{L}\, dx$$

$$\therefore \quad \int_{-L}^L [g_n(x)]^2\, dx = \int_{-L}^L \left(\frac{1+\cos(2n\pi x/L)}{2}\right) dx$$

$$= \frac{1}{2}\left[x + \frac{\sin(2n\pi x/L)}{2n\,\pi/L}\right]_{-L}^L = L \neq 0$$

(c) $\int_{-L}^{L} f_m(x)\cdot g_n(x)\,dx = \int_{-L}^{L} \sin\frac{m\pi x}{L}\cdot\cos\frac{n\pi x}{L}\,dx$

$$= \frac{1}{2}\int_{-L}^{L}\left[\frac{\sin(m+n)\pi x}{L}+\frac{\sin(m-n)\pi x}{L}\right]dx$$

$$= \frac{1}{2}\left[-\frac{\cos\{(m+n)/L\}\pi x}{(m+n)/L}-\frac{\cos\{(m-n)/L\}\pi x}{(m-n)/L}\right]_{-L}^{L}$$

Again two cases arise

Case 1 : If $m \neq n$, then

$$\int_{-L}^{L} f_m(x)\cdot g_n(x)\,dx = 0$$

Case 2 : If $m = n$, then

$$f_n(x)\cdot g_n(x) = [f_n(x)]^2 \text{ or } [g_n(x)]^2$$

and we have already proved above that

$$\int_{-L}^{L}[f_n(x)]^2\,dx = L \text{ and } \int_{-L}^{L}[g_n(x)]^2\,dx = L \text{ and } \int_{-L}^{L} 1\,dx = 2L$$

Hence, the given sequence is orthogonal.

For orthonormality the value of the integral L must be 1. Hence, if each term is devided by $\sqrt{L}$ then the set will be orthonormal. Hence, the set

$\frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}}\sin x, \frac{1}{\sqrt{L}}\cos x, \frac{1}{\sqrt{L}}\sin 2x, \ldots$ is an orthonormal set.

Ex. 7 : Prove that $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = (3x^2-1)/2$ are orthogonal over $(-1, 1)$. **(B.U. 1997, 2002, 03, 04)**

Sol. : We have $\int_{-1}^{1} f_1(x)\cdot f_2(x)\,dx = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$

$$\int_{-1}^{1} f_1(x)\cdot f_3(x)\,dx = \int_{-1}^{1}\frac{1}{2}(3x^2-1)\,dx$$

$$= \frac{1}{2}\left[x^3 - x\right]_{-1}^{1} = 0$$

$$\int_{-1}^{1} f_2(x)\cdot f_3(x)\,dx = \int_{-1}^{1}\frac{x}{2}(3x^2-1)\,dx = \frac{1}{2}\int_{-1}^{1}(3x^3-x)\,dx$$

$$= \frac{1}{2}\left[\frac{3x^4}{4}-\frac{x^2}{2}\right]_{-1}^{1} = 0$$

Further $\int_{-1}^{1} f_1(x)\cdot f_1(x)\,dx = \int_{-1}^{1} 1\cdot 1\cdot dx = [x]_{-1}^{1} = 2 \neq 0$

$$\int_{-1}^{1} f_2(x)\cdot f_2(x)\,dx = \int_{-1}^{1} x\cdot x\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \neq 0$$

$$\int_{-1}^{1} f_3(x)\cdot f_3(x)\,dx = \int_{-1}^{1}\frac{1}{4}(9x^2-6x+1)\,dx$$

$$= \frac{1}{4}\left[3x^3 - 3x^2 + x\right]_{-1}^{1} = 2 \neq 0$$

Hence, the given set is orthogonal over $[-1, 1]$.

Ex. 8 : Show that the functions $f_1(x) = 1$, $f_2(x) = x$ are orthogonal on $(-1, 1)$. Determine the constants a and b such that the function $f_3(x) = -1 + ax + bx^2$ is orthogonal to both f_1 and f_2 on that interval. **(B.U. 2003, 05)**

Sol. : We have already proved the first part above.

Now, if $f_3(x)$ is orthogonal to both $f_1(x)$ and $f_2(x)$ we should have,

(i) $\int_{-1}^{1} f_1(x)\cdot f_3(x)\,dx = 0 \quad \therefore \int_{-1}^{1} 1\cdot(-1+ax+bx^2)\,dx = 0$

$$\therefore \left[-x+\frac{ax^2}{2}+\frac{bx^3}{3}\right]_{-1}^{1} = 0 \quad \therefore \left(-1+\frac{a}{2}+\frac{b}{3}\right)-\left(+1+\frac{a}{2}-\frac{b}{3}\right) = 0$$

$$\therefore -2+\frac{2b}{3} = 0 \quad \therefore b = 3.$$

And (ii) $\int_{-1}^{1} f_2(x)\cdot f_3(x)\,dx = 0 \quad \therefore \int_{-1}^{1} x\cdot(-1+ax+bx^2)\,dx = 0$

$$\therefore \left[\frac{-x^2}{2}+\frac{ax^3}{3}+\frac{bx^4}{4}\right]_{-1}^{1} = 0 \quad \therefore \left(-\frac{1}{2}+\frac{a}{3}+\frac{b}{4}\right)-\left(\frac{1}{2}-\frac{a}{3}+\frac{b}{4}\right) = 0$$

$$\therefore \frac{2a}{3} = 0 \quad \therefore a = 0$$

$$\therefore f_3(x) = 3x^2 - 1$$

Now, $\int_{-1}^{1}[f_3(x)]^2\,dx = \int_{-1}^{1}(3x^2-1)^2\,dx$

$$= \int_{-1}^{1}(9x^2-6x+1)\,dx = \left[3x^3-3x^2+x\right]_{-1}^{1}$$

$$= (3-3+1)-(-3-3-1) = 8 \neq 0$$

The required function $f_3(x) = 3x^2 - 1$.

Ex. 9 : If $f_i(x)$, $i = 1, 2, 3, \ldots$ is a set of orthogonal functions in $[a, b]$ and $g(x) = \sum_{i=1}^{\infty} a_i f_i(x)$, then find a_i. **(M.U. 2003)**

Sol. : We have

$$g(x) = a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) + \ldots\ldots \infty$$

Multiply both sides by $f_1(x)$,

$$\therefore \quad f_1(x)\, g(x) = a_1 [f_1(x)]^2 + a_2 f_1(x) \cdot f_2(x) + a_3 f_1(x)\, f_3(x) + \ldots \infty \qquad \ldots\ldots\ldots\ldots (1)$$

Now, integrate both sides w.r.t. x from a to b.

$$\therefore \quad \int_a^b f_1(x)\, g(x)\, dx = a_1 \int_a^b [f_1(x)]^2 dx + a_2 \int_a^b f_1(x) \cdot f_2(x)\, dx + a_3 \int_a^b f_1(x) \cdot f_3(x)\, dx + \ldots\ldots \infty \qquad \ldots\ldots\ldots\ldots (2)$$

But by definition of orthogonal functions

$$\int_a^b f_m(x)\, f_n(x) = 0 \ \text{ if } m \neq n$$

and $$\int_a^b [f_m(x)]^2 dx \neq 0$$

Hence, on the r.h.s. of (2) all the integrals except the first are zero and the first not zero.

$$\therefore \quad \int_a^b f_1(x)\, g(x)\, dx = a_1 \int_a^b [f_1(x)]^2 dx$$

$$\therefore \quad a_1 = \frac{\int_a^b f_1(x) \cdot g(x)\, dx}{\int_a^b [f_1(x)]^2 dx}$$

Similarly, by multiplying (1) successivaly by $f_2(x)$, $f_3(x)$ ..... and integrating both sides w.r.t. x from a to b, we can obtain the values of a_2, a_3, ...

Thus, in general, we have

$$a_i = \frac{\int_a^b f_i(x)\, g(x)\, dx}{\int_a^b [f_i(x)]^2 dx}.$$

Ex. 10 : If $f(x) = C_1\Phi_1(x) + C_2\Phi_2(x) + C_3\Phi_3(x)$, where C_1, C_2, C_3 constants and Φ_1, Φ_2, Φ_3 are orthonormal sets on (a, b) , show that

$$\int_a^b [f(x)]^2 dx = C_1^2 + C_2^2 + C_3^2 \qquad \textbf{(M.U. 2002)}$$

Sol. : Since, Φ_1, Φ_2, Φ_3 are orthonormal

$$\int_a^b [\Phi_1(x)]^2 dx = \int_a^b [\Phi_2(x)]^2 dx = \int_a^b [\Phi_3(x)]^2 dx = 1 \qquad \ldots\ldots\ldots\ldots (1)$$

and $$\int_a^b \Phi_m(x)\, \Phi_n(x)\, dx = 0 \ \text{ when } m \neq n \qquad \ldots\ldots\ldots\ldots (2)$$

Now $$\int_a^b [f(x)]^2 dx = \int_a^b [C_1\Phi_1(x) + C_2\Phi_2(x) + C_3\Phi_3(x)]^2 dx$$

$$= \int_a^b \Big[C_1^2\{\Phi_1(x)\}^2 + C_2^2(\Phi_2(x))^2 + C_3^2\{\Phi_3(x)\}^2 + 2C_1C_2\,\Phi_1(x)\,\Phi_2(x) + 2C_1C_3\,\Phi_1(x)\,\Phi_3(x) + 2C_2C_3\,\Phi_2(x)\,\Phi_3(x)\Big] dx$$

$$= C_1^2 \int_a^b [\Phi_1(x)]^2 dx + C_2^2 \int_a^b [\Phi_2(x)]^2 dx + C_3^2 \int_a^b [\Phi_3(x)]^2 dx + 2C_1C_2 \int_a^b \Phi_1(x)\,\Phi_2(x)\, dx + 2C_1C_3 \int_a^b \Phi_1(x)\,\Phi_3(x)\, dx + 2C_2C_2 \int_a^b \Phi_1(x)\,\Phi_2(x)\, dx$$

$$= C_1^2 + C_2^2 + C_3^2 \quad \text{by (1) and (2).}$$

EXERCISE

1. Show that the set of functions $\frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}, \ldots\ldots$ form a normal set in the interval $[-\pi, \pi]$. **(B.U. 2003)**

2. Show that the set of functions $\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 3x}{\sqrt{\pi}}, \ldots\ldots$ form a normal set in the interval $[-\pi, \pi]$.

3. Show that the set of functions $\cos x$, $\cos 2x$, $\cos 3x$, ... is a set of orthogonal functions over $[-\pi, \pi]$. Hence, construct a set of orthonormal functions. **(B.U. 1995, 98, 2005)**

4. Prove that $\sin x$, $\sin 2x$, $\sin 3x$, ... is orthogonal on $[0, 2\pi]$ and construct orthonormal set of functions. **(B.U. 1994, 97, 99, 2003, 05)**

5. Is the set of functions $\sin\left(\frac{\pi x}{l}\right)$, $\sin\left(\frac{3\pi x}{l}\right)$, $\sin\left(\frac{5\pi x}{l}\right)$, ... orthogonal over $(0, l)$. **(M.U. 2003)** **[Ans. : Yes]**

6. Is the set of functions $\cos x$, $\cos 3x$, $\cos 5x$, ... orthogonal over $(0, \pi/2)$. **(M.U. 2003)** **[Ans. : No]**

7. Show that the set of functions $\sin x$, sine $2x$, $\sin 3x$, ... is orthogonal on the interval $[0, \pi]$. **(B.U. 1999, 2003)**

8. Show that the set of functions $\cos x$, $\cos 2x$, $\cos 3x$, ... is orthogonal on $[-\pi, \pi]$. **(M.U. 2003)**

9. Show that the set of functions 1, $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, ... is orthogonal on $(0, 2\pi)$ but not on $(0, \pi)$.

How can you convert the set orthonormal on $(0, 2\pi)$? Write down the orthonormal set.

$$\left[\text{Ans. : } \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin 2x, \frac{1}{\sqrt{\pi}}\cos 2x, \ldots\right]$$

10. Show that the following set of functions is orthonormal on $(0, \infty)$

$$\left\{e^{-x/2},\ e^{-x/2}(1-x),\ \frac{1}{2}e^{-x/2}(x^2-4x+2)\right\}$$ **(B.U. 2003, 04)**

4. Fourier Integral Theorem

If $f(x)$ satisfies Dirichlet's conditions (stated in chapter 7) in each finite interval $-l \le x \le l$ and if $f(x)$ is integrable in $-\infty$ to ∞ then Fourier Integral Theorem states that

$$\boxed{f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(s)\cos\omega(s-x)\, d\omega\, ds} \quad \text{.......... (1)}$$

We assume this result without proof.

Note: Unfortunately there is no uniformity in notation and in the use of constants before the integral.

5. Fourier Sine and Cosine Integrals

The above integral can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(s)\{\cos\omega s\cos\omega x\}\, d\omega\, ds$$
$$+\frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(s)\{\sin\omega s\sin\omega x\}\, d\omega\, ds$$

i.e.
$$f(x) = \frac{1}{\pi}\int_0^\infty \cos\omega x\int_{-\infty}^\infty f(s)\cos\omega s\, d\omega\, ds$$
$$+\frac{1}{\pi}\int_0^\infty \sin\omega x\int_{-\infty}^\infty f(s)\sin\omega s\, d\omega\, ds$$

(a) Fourier Cosine Integral

When $f(x)$ is an even function, $f(s)$ will be even but $f(s)\sin\omega s$ will be **odd** function and $f(s)\cos\omega s$ will be **even** function. Hence, the second integral will be zero and we will get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(s)\cos\omega s\, d\omega\, ds} \quad \text{.......... (2)}$$

This is called **Fourier Cosine Integral.**

(b) Fourier Sine Integral

When $f(x)$ is an odd function, $f(s)$ will be odd but $f(s)\sin\omega s$ will be **even** and $f(s)\cos\omega s$ will be **odd**. Hence, the first integral will be zero and we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\infty f(s)\sin\omega s\, d\omega\, ds} \quad \text{.......... (3)}$$

This is called **Fourier Sine Integral.**

Ex. 1 : Express the function $f(x) = \begin{cases} 1 & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$ as Fourier Integral.

(B.U. 1997, 99, 2002, 03)

Hence, evaluate $\displaystyle\int_0^\infty \frac{\sin\omega\sin\omega x}{\cos\omega}\, d\omega$. **(B.U. 1994, 95, 2003)**

Sol. : The Fourier Integral for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(s)\cos\omega(s-x)\, d\omega\, ds$$

[By data $f(s) = 0$ from $-\infty$ to -1, $f(s) = 1$ from -1 to 1 and $f(s) = 0$ from 1 to ∞.]

Hence, $$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-1}^1 1\cdot\cos\omega(s-x)\, d\omega\, ds$$
$$= \frac{1}{\pi}\int_0^\infty \left[\frac{\sin\omega(s-x)}{\omega}\right]_{-1}^1 d\omega$$
$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\omega(1-x) - \sin\omega(-1-x)}{\omega}\, d\omega$$
$$= \frac{1}{\pi}\int_0^\infty \frac{\sin\omega(1+x) + \sin\omega(1-x)}{\omega}\, d\omega$$
$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\omega\cos\omega x}{\omega}\, d\omega$$

$$\therefore \int_0^\infty \frac{\sin\omega\cdot\cos\omega x}{\omega}\, d\omega = \frac{\pi}{2}\cdot f(x)$$
$$= \begin{cases} \dfrac{\pi}{2} \text{ for } f(x) = 1 & \text{when } |x| < 1 \\ 0 \text{ for } f(x) = 0 & \text{when } |x| > 1 \end{cases}$$

At $|x| = 1$ *i.e.* $x = \pm 1$, $f(x)$ is discontinuous and the integral

$$= \frac{\pi}{2}\cdot\frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right]$$
$$= \frac{\pi}{4}[1+0] = \frac{\pi}{4}$$

$$\therefore \int_0^\infty \frac{\sin\omega\cdot\cos\omega x}{\omega}\, d\omega = \begin{cases} \pi/2 & \text{when } |x| < 1 \\ 0 & \text{when } |x| > 1 \\ \pi/4 & \text{when } |x| = 1 \end{cases}$$

Cor. 1 : Putting $x = 1$ in the above result, we get

$$\int_0^{\infty} \frac{\sin \omega \cos \omega}{\omega} d\omega = \frac{\pi}{4}$$

i.e. $$\int_0^{\infty} \frac{\sin 2\omega}{\omega} d\omega = \frac{\pi}{2}$$

Cor. 2 : Putting $x = 0$, in the above result

$$\int_0^{\infty} \frac{\sin \omega}{\omega} d\omega = \frac{\pi}{2}$$

Ex. 2 : Express the function $f(x) = \begin{cases} \sin x, & 0 < x \le \pi \\ 0, & x < 0,\ x > \pi \end{cases}$

as Fourier Integral and prove that

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\sin \omega x + \cos[\omega(\pi - x)]}{1 - \omega^2} d\omega$$ **(M.U. 2001)**

Sol. : The Fourier integral for $f(x)$ is

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(s) \cos \omega (s - x)\, d\omega\, ds$$

$$= \frac{1}{\pi} \int_0^{\infty} \int_0^{\pi} \sin s \cos \omega (s - x)\, d\omega\, ds$$

$$= \frac{1}{2\pi} \int_0^{\infty} \int_0^{\pi} 2 \sin s \cos \omega (s - x)\, d\omega\, ds$$

$$= \frac{1}{2\pi} \int_0^{\infty} \int_0^{\pi} [\sin(s + \omega s - \omega x) + \sin(s - \omega s + \omega x)]\, d\omega\, ds$$

$$= \frac{1}{2\pi} \int_0^{\infty} \left[-\frac{\cos(s + \omega s - \omega x)}{1 + \omega} - \frac{\cos(s - \omega s + \omega x)}{1 - \omega} \right]_0^{\pi} d\omega$$

$$= \frac{1}{2\pi} \int_0^{\infty} \left[-\frac{\cos(\pi + \pi\omega - \omega x)}{1 + \omega} - \frac{\cos(\pi - \pi\omega + \omega x)}{1 - \omega} + \frac{\cos \omega x}{1 + \omega} + \frac{\cos \omega x}{1 - \omega} \right] d\omega$$

$$= \frac{1}{2\pi} \int_0^{\infty} \left[\frac{\cos(\pi\omega - \omega x)}{1 + \omega} + \frac{\cos(-\pi\omega + \omega x)}{1 - \omega} + \frac{\cos \omega x}{1 + \omega} + \frac{\cos \omega x}{1 - \omega} \right] d\omega$$

$$= \frac{1}{2\pi} \int_0^{\infty} \left[\left(\frac{1}{1 + \omega} \right) \{\cos \omega x + \cos \omega (\pi - x)\} + \left(\frac{1}{1 - \omega} \right) \{\cos \omega x + \cos \omega (\pi - x)\} \right] d\omega$$

$$= \frac{1}{2\pi} \int_0^{\infty} \left(\frac{1}{1 + \omega} + \frac{1}{1 - \omega} \right) \{\cos \omega x + \cos \omega (\pi - x)\}\, d\omega$$

$$= \frac{1}{2\pi} \int_0^{\infty} \frac{2}{(1 - \omega^2)} \{\cos \omega x + \cos \omega (\pi - x)\}\, d\omega$$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\frac{\cos \omega x + \cos \omega (\pi - x)}{(1 - \omega^2)} \right] d\omega$$

Putting $x = \pi/2$, we get,

$$\sin \frac{\pi}{2} = \frac{1}{\pi} \int_0^{\infty} \frac{\left[\cos \frac{\pi\omega}{2} + \cos \frac{\pi\omega}{2} \right]}{(1 - \omega^2)} d\omega$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{\cos \pi\omega / 2}{1 - \omega^2} d\omega$$

Note : Unfortunately there is no uniformity in the notation of Fourier Integral. Some authors use λ or α in place of ω and t in place of s.

Ex. 3 : Express the function

$$f(x) = \begin{cases} -e^{kx} & \text{for } x < 0 \\ e^{-kx} & \text{for } x > 0 \end{cases}$$

as Fourier Integral and hence, prove that

$$\int_0^{\infty} \frac{\omega \sin \omega x}{\omega^2 + k^2} d\omega = \frac{\pi}{2} e^{-kx} \text{ if } x > 0,\ k > 0$$ **(M.U. 2002)**

Sol. : Since, the given function $f(x)$ is an odd function we use (3)

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \omega x \int_0^{\infty} f(s) \sin \omega s\, d\omega\, ds$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \omega x \int_0^{\infty} e^{-ks} \sin \omega s\, d\omega\, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \sin \omega x \left[\frac{1}{k^2 + \omega^2} e^{-ks} (-k \sin \omega s - \omega \cos \omega s) \right]_0^{\infty} d\omega$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \omega x \cdot \frac{\omega}{k^2 + \omega^2} d\omega$$

$$\therefore \int_0^{\infty} \frac{\omega \sin \omega x}{\omega^2 + k^2} d\omega = \frac{\pi}{2} f(x) = \frac{\pi}{2} e^{-kx} \text{ if } x > 0$$

Ex. 4 : Using Fourier Cosine Integral prove that

$$e^{-x} \cos x = \frac{2}{\pi} \int_0^{\infty} \frac{(\omega^2 + 2)}{(\omega^4 + 4)} \cdot \cos \omega x\, d\omega$$ **(M.U. 2002, 05)**

Sol. : By Fourier cosine integral formula

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty f(s) \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty e^{-s} \cos s \cdot \cos \omega s \, d\omega \, ds$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \int_0^\infty e^{-s}[\cos(\omega+1)s + \cos(\omega-1)s] \, d\omega \, ds$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \left[\frac{1}{1+(\omega+1)^2} \cdot e^{-s}\{-\cos(\omega+1)s + (\omega+1)\sin(\omega+1)s\}_0^\infty + \frac{1}{1+(\omega-1)^2} \cdot e^{-s}\{-\cos(\omega-1)s + (\omega-1)\sin(\omega-1)s\}_0^\infty\right]$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \left[\frac{1}{1+(\omega+1)^2} + \frac{1}{1+(\omega-1)^2}\right] d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \left[\frac{\omega^2 - 2\omega + 2 + \omega^2 + 2\omega + 2}{\{(\omega^2+2)+2\omega\}\{(\omega^2+2)+2\omega\}}\right] d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \cdot \frac{2(\omega^2+2)}{(\omega^2+2)^2 - 4\omega^2} \cdot d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \cos \omega x \cdot \frac{(\omega^2+2)}{(\omega^4+4)} d\omega$$

Ex. 5 : Find Fourier Integral representation for

$$f(x) = \begin{cases} 1 - x^2 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$ **(B.U. 1998, 2003)**

Sol. : By data $f(s) = 0$ from $-\infty$ to -1, $f(s) = 1 - s^2$ from -1 to 1 and $f(s) = 0$ from 1 to ∞.

Also $f(-s) = 1 - (-s)^2 = 1 - s^2$

$= f(s)$ from -1 to 1

Hence, $f(s)$ is an even function and we use (2).

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty f(s) \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[\int_0^1 (1 - s^2) \cos \omega s \, ds\right] d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[(1-s^2)\left(\frac{\sin \omega s}{\omega}\right) - \left(-\frac{\cos \omega s}{\omega^2}\right)(-2s) + \left(-\frac{\sin \omega s}{\omega^3}\right)(-2)\right]_0^1 d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[0 - \frac{2\cos \omega}{\omega^2} + \frac{2 \sin \omega}{\omega^3}\right] d\omega$$

$$= \frac{4}{\pi}\int_0^\infty \frac{\sin \omega - \omega \cos \omega}{\omega^3} \cdot \cos \omega x \, d\omega$$

Ex. 6 : Find Fourier integral representation of

$$f(x) = \begin{cases} e^{ax} & x \le 0,\ a > 0 \\ e^{-ax} & x \ge 0,\ a > 0 \end{cases}$$ **(B.U. 1996, 97, 2002)**

Hence, show that

$$\int_0^\infty \frac{\cos \omega x}{\omega^2 + a^2} d\omega = \frac{\pi}{2a} e^{-ax}, \quad x > 0,\ a > 0.$$

Sol. : Since, $f(x)$ is an even function we use (2).

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty f(s) \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty e^{-as} \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[\frac{1}{a^2+\omega^2} \cdot e^{-as}(-a \cos \omega s + \omega \sin \omega s)\right]_0^\infty d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \cdot \frac{a}{a^2+\omega^2} d\omega$$

$$\therefore \quad \int_0^\infty \frac{\cos \omega x}{a^2+\omega^2} d\omega = \frac{\pi}{2a} f(x) = \frac{\pi}{2a} e^{-ax}, \ x > 0,\ a > 0$$

Ex. 7 : Find Fourier Integral representation of

$f(x) = x, \quad 0 < x < a$

$= 0, \quad x > a$

$f(-x) = f(x)$ **(B.U. 1995)**

Sol. : Since, $f(x)$ is even function we use (2).

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^\infty f(s) \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \int_0^a s \cos \omega s \, d\omega \, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[\frac{s(\sin \omega s)}{\omega} - \int \frac{\sin \omega s}{\omega}(1) \cdot ds\right]_0^a d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x \left[\frac{s(\sin \omega s)}{\omega} + \frac{\cos \omega s}{\omega^2}\right]_0^a d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x\left[\frac{a\sin a\omega}{\omega}+\frac{\cos a\omega}{\omega^2}-\frac{1}{\omega^2}\right]d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos \omega x\left(\frac{a\omega\sin a\omega+\cos a\omega-1}{\omega^2}\right)d\omega$$

Ex. 8 : Express the function

$$f(x) = \begin{cases} \pi/2 & \text{for } 0 < x < \pi \\ 0 & \text{for } x > \pi \end{cases}$$

as Fourier Sine Integral. **(B.U. 1998)**

Hence, show that

$$\int_0^\infty \frac{1-\cos \pi\omega}{\omega}\sin \omega x\, d\omega = \frac{\pi}{2} \text{ when } 0 < x < \pi.$$

Sol. : Fourier Sine Integral by (3), page 8-23 is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin \omega x\int_0^\pi \frac{\pi}{2}\cdot \sin \omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\cdot\frac{\pi}{2}\left[-\frac{\cos \omega s}{\omega}\right]_0^\pi d\omega$$

$$= \int_0^\infty \sin \omega x\left[\frac{-\cos\pi\omega+1}{\omega}\right]d\omega$$

$$= \int_0^\infty \frac{1-\cos\pi\omega}{\omega}\cdot\sin\omega x\, d\omega$$

$$\therefore \int_0^\infty \frac{1-\cos\pi\omega}{\omega}\cdot\sin\omega x\cdot d\omega = f(x) = \frac{\pi}{2} \text{ when } 0 < x < \pi.$$

Ex. 9 : Find Fourier Sine integral representation for $f(x) = \frac{e^{-ax}}{x}$. **(B.U. 2004)**

Sol. : By (3) Fourier Sine integral is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\infty f(s)\sin\omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\infty \frac{e^{-as}}{s}\sin\omega s\, ds\, d\omega$$

To evaluate $\int_0^\infty \frac{e^{-as}}{s}\sin\,\omega s\, ds$

we use the rule of differentiation under the integral sign

Let $\quad I = \int_0^\infty \frac{e^{-as}}{s}\cdot\sin\omega s\, ds$

$$\therefore \frac{dI}{d\omega} = \int_0^\infty \frac{\partial}{\partial\omega}\left(\frac{e^{-as}}{s}\cdot\sin\omega s\right)ds$$

$$= \int_0^\infty \frac{e^{-as}}{s}\cdot(\cos\omega s)\cdot s\, ds = \int_0^\infty e^{-as}\cos\omega s\, ds$$

$$= \frac{1}{a^2+\omega^2}\left[e^{-as}(-a\cos\omega s+\omega\sin\omega s)\right]_0^\infty$$

$$= \frac{1}{a^2+\omega^2}(a)$$

$$\therefore \frac{dI}{d\omega} = \frac{a}{a^2+\omega^2}$$

Integrating w.r.t. ω,

$$I = a\cdot\frac{1}{a}\tan^{-1}\frac{\omega}{a}+C$$

$$\therefore \int_0^\infty \frac{e^{-as}}{s}\cdot\sin\omega s\, ds = \tan^{-1}\frac{\omega}{a}+C$$

To find C, we put $\omega = 0$.

$$\therefore \quad 0 = 0 + C \quad \therefore C = 0$$

$$\therefore \int_0^\infty \frac{e^{-as}}{s}\cdot\sin\omega s\, ds = \tan^{-1}\frac{\omega}{a}$$

Hence, $f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x\cdot\tan^{-1}\frac{\omega}{a}\, d\omega$

Ex. 10 : Find Fourier Sine integral of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

(B.U. 1999)

Sol. : Fourier Sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x\int_0^\infty f(s)\sin\omega s\cdot d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left[\int_0^1 s\sin\omega s\, ds+\int_1^2 (2-s)\sin\omega s\, ds+\int_2^\infty 0\cdot\sin\omega s\, ds\right]d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left\{\left[s\left(-\frac{\cos\omega s}{\omega}\right)-\left(-\frac{\sin\omega s}{\omega^2}\right)(1)\right]_0^1\right.$$

$$\left.+\left[(2-s)\left(-\frac{\cos\omega s}{\omega}\right)-\left(-\frac{\sin\omega s}{\omega^2}\right)(-1)\right]_1^2\right\}d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left\{\left[-\frac{\cos\omega}{\omega}+\frac{\sin\omega}{\omega^2}\right]\right.$$

$$\left.+\left[0-\frac{\sin 2\omega}{\omega^2}+\frac{\cos\omega}{\omega}+\frac{\sin\omega}{\omega^2}\right]\right\}d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\cdot\frac{(2\sin\omega-\sin 2\omega)}{\omega^2}\,d\omega$$

Ex. 11 : Find Fourier cosine integral for

$$f(x)=\begin{cases}1-x^2, & 0\le x\le 1\\ 0, & x>1\end{cases}$$

Hence, evaluate $\int_0^\infty\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\cdot dx$. **(M.U. 2003)**

Sol. : Fourier cosine integral for $f(x)$ is

$$f(x)=\frac{2}{\pi}\int_0^\infty\cos\omega x\left\{\int_0^\infty f(t)\cos\omega t\,dt\right\}d\omega$$

$$=\frac{2}{\pi}\int_0^\infty\cos\omega x\left\{\int_0^\infty(1-t^2)\cos\omega t\,dt\right\}d\omega$$

$$=\frac{2}{\pi}\int_0^\infty\cos\omega x\left\{\left[(1-t^2)\frac{\sin\omega t}{\omega}-(-2t)\left(-\frac{\cos\omega t}{\omega^2}\right)\right.\right.$$

$$\left.\left.+(-2)\left(-\frac{\sin\omega t}{\omega^3}\right)\right]_0^1\right\}d\omega$$

$$=\frac{2}{\pi}\int_0^\infty\cos\omega x\left\{-2\cdot\frac{\cos\omega}{\omega^2}+\frac{2\sin\omega}{\omega^3}\right\}d\omega$$

$$1-x^2=\frac{4}{\pi}\int_0^\infty\cos\omega x\left(\frac{\sin\omega-\omega\cos\omega}{\omega^3}\right)d\omega$$

Now put $x=1/2$,

$$\therefore\quad \frac{3\pi}{16}=\int_0^\infty\frac{\sin\omega-\omega\cos\omega}{\omega^3}\cos\frac{\omega}{2}\cdot d\omega$$

EXERCISE

1. Express the function

$$f(x)=\begin{cases}1 & \text{for } 0\le x<1\\ 0 & \text{for } x>1\end{cases}$$

as a Fourier Cosine Integral and hence, show that

$$\int_0^\infty\frac{\sin\omega\cdot\cos\omega x}{\omega}\,d\omega=\frac{\pi}{2}\quad\text{if } 0\le x<1$$

Also show that the integral is equal to $\pi/4$ for $x=1$ and zero for $x>1$.

$\left(\textbf{Hint : } f(x)=\frac{2}{\pi}\int_0^\infty\cos\omega x\int_0^1 1\cdot\cos\omega s\,d\omega\,ds=\frac{2}{\pi}\int_0^\infty\frac{\cos\omega x\cdot\sin\omega}{\omega}\,d\omega\right)$

2. Find the Fourier Integral representation of

$$f(x)=\begin{cases}0, & x<0\\ 1/2, & x=0\\ e^{-x}, & x>0\end{cases}$$

$$\left(\textbf{Hint : } f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^0 0\,d\omega\,ds+\int_0^\infty\int_0^\infty e^{-s}\cos\omega(s-x)\,d\omega\,ds\right.$$

$$=\frac{1}{\pi}\int_0^\infty\left\{\cos\omega x\int_0^\infty e^{-s}\cos\omega s\,ds+\sin\omega s\int_0^\infty e^{-s}\sin\omega s\,ds\right\}d\omega$$

$$=\frac{1}{\pi}\int_0^\infty\frac{\cos\omega x+\omega\sin\omega x}{1+\omega^2}\,d\omega$$

$$\left.\text{when } x=0,\ f(0)=\frac{1}{\pi}\int_0^\infty\frac{1}{1+\omega^2}\,d\omega=\frac{1}{\pi}\left[\tan^{-1}\omega\right]_0^\infty=\frac{1}{2}\right)$$

3. Express the function

$$f(x)=\begin{cases}\sin x, & 0\le x\le\pi\\ 0, & x>\pi\end{cases}$$

as Fourier Sine Integral and evaluate

$$\int_0^\infty\frac{\sin\omega x\cdot\sin\pi\omega}{1-\omega^2}\,d\omega$$ **(B.U. 2000)**

$$\left(\textbf{Hint : } f(x)=\frac{2}{\pi}\int_0^\infty\sin\omega x\int_0^\infty\sin s\sin\omega s\,d\omega\,ds\right.$$

$$=\frac{2}{\pi}\int_0^\infty\sin\omega x\left[-\frac{1}{2}\right]\int_0^\infty\left[\cos s(1+\omega)-\cos s(1-\omega)\right]d\omega\,ds$$

$$\left.=\frac{2}{\pi}\int_0^\infty\sin\omega x\left(-\frac{1}{2}\right)\cdot\left(-2\cdot\frac{\sin\pi\omega}{1-\omega^2}\right)d\omega=\frac{2}{\pi}\int_0^\infty\frac{\sin\omega x\cdot\sin\pi\omega}{1-\omega^2}\,d\omega\right)$$

4. Express the function

$$f(x)=e^{-x}-e^{-2x},\quad x\ge 0$$

as Fourier Sine Integral and evaluate

$$\int_0^\infty\frac{\omega\sin\omega x}{(1+\omega^2)(4+\omega^2)}\,d\omega.$$ $\left[\textbf{Ans. : } \frac{\pi}{6}(e^{-x}-e^{-2x})\right]$

5. Express the function $f(x) = e^{-x}$ as Fourier Sine integral $(x \geq 0)$

and show that $\int_0^{\infty} \frac{\cos \omega x}{1+\omega^2} d\omega = \frac{\pi}{2} \cdot e^{-x}$.

6. Express $f(x) = \begin{cases} 0 & \text{for } x < 0 \\ e^{-x} & \text{for } x \geq 0 \end{cases}$

as a Fourier Integral and show that

$$\int_0^{\infty} \frac{\cos \omega x + \omega \sin \omega x}{1+\omega^2} d\omega = \begin{cases} 0 & \text{for } x < 0 \\ 1/2 & \text{for } x = 0 \\ \pi e^{-x} & \text{for } x > 0 \end{cases}$$

(**Hint :** Foe second result put $x = 0$ in the integral, then

$$f(0) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{1+\omega^2} d\omega = \frac{1}{\pi} \left[\tan^{-1} \omega\right]_0^{\infty} = \frac{1}{2}.)$$

7. Express $f(x) = \frac{\pi}{2} e^{-x} \cos x$ for $x > 0$

as Fourier Sine integral and show that

$$\int_0^{\infty} \frac{\omega^3 \sin \omega x}{\omega^4 + 4} d\omega = \frac{\pi}{2} e^{-x} \cos x$$ **(M.U. 2002)**

(**Hint :** Use $2 \sin \omega x \cos x = [\sin (\omega + 1) x + \sin (\omega - 1) x]$ and

$$\int e^{ax} \sin bx \, dx = \frac{1}{a^2 + b^2} e^{ax} [a \sin bx - b \cos bx])$$

8. Express $f(x) = e^{-kx}$ $(k > 0)$

as Fourier Sine and Cosine Integral and show respectively that

(i) $\int_0^{\infty} \frac{\omega \sin \omega x}{k^2 + \omega^2} d\omega = \frac{\pi}{2} e^{-kx}$ (ii) $\int_0^{\infty} \frac{\omega \cos \omega x}{k^2 + \omega^2} d\omega = \frac{\pi}{2k} e^{-kx}$

9. Express the following function as Fourier Integral

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

$$\left[\textbf{Ans. :} \ f(x) = \frac{2}{\pi} \int_0^{\infty} \left(1 - \frac{2}{\omega^2} \sin \omega + \frac{2}{\omega} \cos \omega\right) \frac{\cos \omega x}{\omega} d\omega\right]$$

Theory

1. State Fourier Integral Theorem.
2. Define Fourier Sine and Cosine Integral . **(B.U. 1998, 2005)**

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5

LAPLACE TRANSFORMS - I

1. Introduction

By using a particular type of definite integral as an operator a new function can be defined. One such operator is called **Laplace Transform.** Laplace transform changes a function of one variable denoted by t into a function of another variable denoted by s.

2. Definition

If $f(t)$ is a function of t satisfying certain conditions, then the definite integral

$$\boxed{\Phi(s) = \int_0^{\infty} e^{-st} \cdot f(t) \, dt}$$ ...

when it exists, is called the **Laplace Transform** of $f(t)$ and is writte
Thus,

$$\boxed{L[f(t)] = \int_0^{\infty} e^{-st} \cdot f(t) \, dt}$$

/e is

/unction

Laplace transform exists if the integral on th
convergent.

of s.

There is one to one correspondence betwee
of t and $\Phi(s)$ which is a function of s.

Note : Some authors use $F(s)$ in place of Φ

Ex. 1 : Find the Laplace transform $e^{\cdot} \cdot f(t)\,dt$
$f(t) = t$, for $0 < t < 4$ and $f(t$

» 1.
. 2003)

Sol. : $L[f(t)] = \int_0^{\infty} e^{-st} \cdot f(t) \, dt = \int_0^4$

$= \int_0^4 e^{-st} \cdot t \, dt + \int^{\infty} \left[\frac{e^{-st}}{-s}\right]_4^{\infty}$

$= \left[(t) \cdot \left(\frac{e^{-st}}{-s}\right.\right.$

$$= -\frac{4}{s} \cdot e^{-4s} - \frac{1}{s^2} \cdot e^{-4s} + \frac{1}{s^2} + \frac{5}{s} \cdot e^{-4s}$$

$$= \frac{1}{s^2} + \left(\frac{1}{s} - \frac{1}{s^2}\right) \cdot e^{-4s}$$

Ex. 2 : Find the Laplace transform of $f(t)$, where

(1) $f(t) = a,\ 0 < t < b$ and $f(t) = 0,\ t > b.$

(2) $f(t) = \cos(t - \alpha),\ t > \alpha$ and $f(t) = 0,\ t < \alpha.$

Sol. : (1) $L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\,dt = \int_0^b e^{-st} \cdot f(t)\,dt + \int_b^\infty e^{-st} \cdot f(t)\,dt$

$$= \int_0^b e^{-st} \cdot a\,dt + \int_b^\infty e^{-st} \cdot 0\,dt = a\int_0^b e^{-st}dt$$

$$= a\left[\frac{-e^{-st}}{s}\right]_0^b = \frac{a}{s}(1 - e^{-bt})$$

(2) $L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\,dt = \int_0^\alpha 0 \cdot dt + \int_\alpha^\infty e^{-st} \cdot \cos(t - \alpha)\,dt$

But $\int e^{ax} \cos bx\,dx = \frac{1}{a^2 + b^2} e^{ax}(a\cos bx + b\sin bx)$

$$\therefore\quad L[f(t)] = \left[\frac{1}{s^2 + 1} \cdot e^{-st}\{-s\cos(t - \alpha) + \sin(t - \alpha)\}\right]_\alpha^\infty$$

$$= \frac{se^{-\alpha s}}{s^2 + 1}$$

Ex. 3 : Find the Laplace transform of

$f(t) = \cos t$, for $0 < t < \pi$ and $f(t) = \sin t$, for $t > \pi$. **(M.U. 1993, 2002)**

Sol. : $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st} \cos t\,dt + \int_\pi^\infty e^{-st} \sin t\,dt.$

But $\int e^{ax} \cos bx\,dx = \frac{1}{(a^2 + b^2)} \cdot e^{ax}(a\cos bx + b\sin bx)$ and

$$\int e^{ax} \sin bx\,dx = \frac{1}{(a^2 + b^2)} e^{ax}(a\sin bx - b\cos bx) \qquad \text{.......... (A)}$$

$$\therefore\quad L\,f(t) = \frac{1}{s^2 + 1} \cdot \left[e^{-st}(-s\cos t + \sin t)\right]_0^\pi + \frac{1}{s^2 + 1} \cdot \left[e^{-st}(-s\sin t - \cos t)\right]_\pi^\infty$$

$$= \frac{1}{s^2 + 1} \cdot \left[e^{-s\pi}(s) - (-s)\right] + \frac{1}{s^2 + 1} \cdot \left[-e^{-s\pi}\right]$$

$$= \frac{1}{s^2 + 1} \cdot \left[s + (s - 1)\,e^{-s\pi}\right]$$

Ex. 4 : Using the definition of Laplace transform, prove that

$$L\left[L_n(t)\right] = \frac{(s - 1)^n}{s^{n+1}} \quad \text{where, } L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(e^{-x}x^n).$$

Sol. : By definition

$$L\left[L_n(t)\right] = \int_0^\infty e^{-st}\,\frac{e^t}{n!}\,\frac{d^n}{dt^n}(e^{-t}\,t^n)\,dt$$

$$= \int_0^\infty \frac{e^{-(s-1)t}}{n!}\,\frac{d^n}{dt^n}(e^{-t}\,t^n)\,dt$$

$$= \left[\frac{e^{-(s-1)t}}{n!} \cdot \frac{d^{n-1}}{dt^{n-1}}(e^{-t}\,t^n)\right]_0^\infty$$

$$- \int_0^\infty - \frac{d^{n-1}}{dt^{n-1}}(e^{-t}\,t^n) \cdot (s - 1)\,\frac{e^{-(s-1)t}}{n!}\,dt$$

$$= (s - 1)\int_0^\infty \frac{e^{-(s-1)t}}{n!} \cdot \frac{d^{n-1}}{dt^{n-1}}(e^{-t}\,t^n)\,dt$$

Integrating in this way n times,

$$L\left[L_n(t)\right] = \frac{(s - 1)^n}{n!}\int_0^\infty e^{-(s-1)t} \cdot e^{-t} \cdot t^n \cdot dt$$

$$= \frac{(s - 1)^n}{n!}\int_0^\infty e^{-st} \cdot t^n \cdot dt$$

$$= \frac{(s - 1)^n}{n!}\,L[t^n]$$

$$= \frac{(s - 1)^n}{n!} \cdot \frac{n!}{s^{n+1}} = \frac{(s - 1)^n}{s^{n+1}} \quad \text{(See 10 page 3-6)}$$

EXERCISE

Find the Laplace transform of $f(t)$, where,

1. $f(t) = 3,\ 0 < t < 5$; $\qquad f(t) = 0,\ t > 5.$
2. $f(t) = (t - 1)^2,\ t > 1$; $\qquad f(t) = 0,\ 0 < t < 1.$
3. $f(t) = (t - 1)^3,\ t > 1$; $\qquad f(t) = 0,\ 0 < t < 1.$
4. $f(t) = t,\ 0 < t < 1/2$; $\qquad f(t) = t - 1,\ 1/2 < t < 1$; $f(t) = 0,\ t > 1.$ **(M.U. 2003)**
5. $f(t) = \sin 2t,\ 0 < t < \pi$; $\qquad f(t) = 0,\ t > \pi.$
6. $f(t) = 0,\ 0 \le t \le 4$; $\qquad f(t) = t,\ 1 < t < 2,\ f(t) = 0,\ t > 2.$

7. $f(t) = t,\ 0 < t < 3$; $f(t) = 6,\ t > 3$.
8. $f(t) = \cos t,\ 0 < t < 2\pi$; $f(t) = 0,\ t > 2\pi$. **(M.U. 2002, 04)**
9. $f(t) = t^2,\ 0 < t < 1$; $f(t) = 1,\ t > 1$. **(M.U. 2003)**
10. $f(t) = (t-a)^3,\ t > a$; $f(t) = 0,\ t < a$.
11. $f(t) = t,\ 0 < t < a$; $f(t) = b,\ t > a$.
12. $f(t) = 0,\ 0 < t < \pi$; $f(t) = \sin^2(t-\pi),\ t > \pi$. **(M.U. 2003)**

[**Ans.:** (1) $\frac{3}{s}(1-e^{-5s})$, (2) $\frac{2e^{-s}}{s^3}$, (3) $\frac{6}{s^4}e^{-s}$, (4) $\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s/2}}{s}$,

(5) $\frac{2(1-e^{-\pi s})}{s^2+4}$, (6) $\left(\frac{1}{s^2}+\frac{1}{s}\right)\cdot e^{-s} - \left(\frac{1}{s^2}+\frac{2}{s}\right)\cdot e^{-2s}$,

(7) $\frac{1}{s^2} + \left(\frac{3}{s} - \frac{1}{s^2}\right)e^{-3s}$, (8) $(1-e^{-2\pi s})\frac{s}{s^2+1}$, (9) $-\left(\frac{e^{-s}+1}{s}\right)$,

(10) $\frac{6}{s^4}e^{-as}$, (11) $\frac{1}{s^2} + \left[\frac{(b-a)}{s} - \frac{1}{s^2}\right]e^{-as}$, (12) $\frac{e^{-\pi s}}{2s} - \frac{s\cdot e^{-\pi s}}{s^2+4}$.]

(**Hint :** For integration, use generalised rule of integration by parts.)

3. Linearity Property

If k_1 and k_2 are constants then,

$$\boxed{L[k_1 f_1(t) + k_2 f_2(t)] = k_1 L[f_1(t)] + k_2 L[f_2(t)]} \qquad \text{.......... (3)}$$

The result can be easily proved by using the above definition.

4. Laplace Transforms Of Standard Functions

By using the definition, we find below Laplace Transforms of some standard functions.

(i) $$\boxed{L(e^{at}) = \frac{1}{s-a}(s > a)} \qquad \text{.......... (4)}$$

Proof : We have $L(e^{at}) = \int_0^\infty e^{-st}\cdot e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\cdot dt$

$$= \frac{1}{-(s-a)}\left[e^{-(s-a)t}\right]_0^\infty = \frac{1}{s-a} \qquad \text{.......... (A)}$$

Cor. 1 : If $a = 0$, $(s > 0)$ $\boxed{L(1) = \frac{1}{s}}$ .... (5)

Cor. 2 : Changing the sign of a, $\boxed{L(e^{-at}) = \frac{1}{s+a}}$ .... (6)

Cor. 3 : Since $c^a = e^{a\log c}$, $(c > 0,\ s > a\log c)$

$$\boxed{L(c^{at}) = \frac{1}{s - a\log c}}$$

Note : Note that the integral in (A) is defined only if $(s-a) > 0$ *i.e.* $s > a$.

(ii) $\boxed{L(\sin at) = \frac{a}{s^2+a^2}}$ and $\boxed{L(\cos at) = \frac{s}{s^2+a^2}(s > 0)}$

.... (7)

Proof : Consider,

$$L(\cos at + i\sin at) = L(e^{iat}) = \frac{1}{s-ai} \text{ by (i)} = \frac{s+ai}{s^2+a^2}$$

Equating real and imaginary parts we get (4).

Note : The above transforms (4) can also be obtained directly by using the definition.

(iii) $\boxed{L(\sinh at) = \frac{a}{s^2-a^2}}$ and $\boxed{L(\cosh at) = \frac{s}{s^2-a^2}(s > |a|)}$

.... (8)

Proof : We have, $L(\sinh t) = L\left(\frac{e^{at}-e^{-at}}{2}\right)$

$$= \frac{1}{2}\left\{L(e^{at}) - L(e^{-at})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} \text{ by (i)}$$

$$= \frac{a}{s^2-a^2}$$

Similarly, we can prove $L(\cosh t) = \frac{s}{s^2-a^2}$

(iv) $\boxed{L(t^n) = \frac{\overline{|(n+1)}}{s^{n+1}}}$ $[\,(n+1) > 0 \text{ and } s > 0\,]$ .... (9)

Proof : We have, $L(t^n) = \int_0^\infty e^{-st}\cdot t^n\,dt$. Put $z = st$.

$$= \int_0^\infty e^{-z}\cdot\frac{z^n}{s^n}\cdot\frac{dz}{s} = \frac{\overline{|n+1}}{s^{n+1}}$$

Ex. : Find $L\left[\frac{1}{\sqrt{\pi t}}\right]$.

Sol. : By definition $L\left[\frac{1}{\sqrt{t}}\right] = \int_0^{\infty} e^{-st} \cdot t^{-1/2}\, dt$

Putting $st = z,\ s\,dt = dz$

$$L\left[\frac{1}{\sqrt{t}}\right] = \int_0^{\infty} e^{-z}\left(\frac{z}{s}\right)^{-1/2} \cdot \frac{dz}{s}$$

$$= \int_0^{\infty} e^{-z} \cdot z^{-1/2} \cdot s^{1/2} \cdot s^{-1}\, dz$$

$$= s^{-1/2}\int_0^{\infty} e^{-z} \cdot z^{-1/2}\, dz$$

$$= s^{-1/2}\left|\frac{1}{2}\right. = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\therefore\quad L\left[\frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{t}}\right] = \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{s}} = \frac{1}{\sqrt{s}}.$$

Cor. 4 : If n is a positive integer, $\left|\overline{n+1}\right. = n!$

$$\boxed{L(t^n) = \frac{n!}{s^{n+1}} \text{ if } n \text{ is a +ve integer.}} \qquad \text{.......... (10)}$$

e.g. $L(1) = \frac{1}{s},\ L(t) = \frac{1}{s^2},\ L(t^2) = \frac{2}{s^3},\ L(t^3) = \frac{6}{s^4}$

Note : For existence of the above Laplace transforms the conditions given in the brackets are necessary.

The conditions for existence of Laplace transform are - If $f(t)$ is continuous and $\lim_{t \to \infty}\left\{e^{-at} f(t)\right\}$ is finite, then the Laplace transform of $f(t)$ *i.e.* $\int_0^{\infty} e^{-st} f(t)\, dt$ exists for $s > a$.

EXERCISE

Write down the Laplace transforms of the following.

1. t^2, **2.** $t^{3/2}$, **3.** t^4, **4.** $t^{1/2}$, **5.** 1, **6.** e^{2t}, **7.** e^{-4t},

8. $\sin 2t$, **9.** $\cos 3t$, **10.** $\sin 5t$, **11.** $\cos t$, **12.** 10^{2t},

13. $\sin h\, 3t$, **14.** $\cos h\, 2t$, **15.** 5^{3t}, **16.** $1/\sqrt{\pi t}$.

[**Ans. :** Answers not given for obvious reason.]

Ex. 1 : Find the Laplace transforms of

(i) $4t^2 + \sin 3t + e^{2t}$ **(ii)** $(\sin 2t - \cos 2t)^2$

(iii) $\cos h^2 4t$ **(M.U. 2004)**

Sol. : (i) $L[4t^2 + \sin 3t + e^{2t}] = 4L(t^2) + L(\sin 3t) + L(e^{2t})$

$$= 4 \cdot \frac{2}{s^3} + \frac{3}{s^2 + 3^2} + \frac{1}{s-2}$$

(ii) $L[(\sin 2t - \cos 2t)^2] = L(1 - 2\sin 2t \cos 2t)$

$$= L(1 - \sin 4t) = L(1) - L(\sin 4t) = \frac{1}{s} - \frac{4}{s^2 + 4^2}$$

(iii) $L[\cos h^2 4t] = L[\frac{1}{2}(1 + \cos h\, 8t)]$

$$= \frac{1}{2}[L(1) + L(\cos h 8t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 - 8^2}\right]$$

Ex. 2 : Find the Laplace transforms of

(i) $\sin(\omega t + \alpha)$. **(ii)** $t^2 - e^{-2t} + \cos h^2 3t$.

Sol. : (i) $L[\sin(\omega t + \alpha)] = L[\sin \omega t \cos \alpha + \cos \omega t \sin \alpha]$

$$= \cos \alpha\, L(\sin \omega t) + \sin \alpha\, L(\cos \omega t)$$

$$= \cos \alpha \cdot \frac{\omega}{s^2 + \omega^2} + \sin \alpha \cdot \frac{s}{s^2 + \omega^2}.$$

(ii) $L[t^2 - e^{-2t} + \cos h^2 3t]$

$$= L[t^2] - L[e^{-2t}] + L\frac{1}{2}[1 + \cos h\, 6t]$$

$$= \frac{2}{s^3} - \frac{1}{s+2} + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 - 6^2}\right]$$

Ex. 3 : Find the Laplace transform of the following.

(i) $\sin^5 t$ **(M.U. 1993, 2003, 05)** **(ii)** $\sin h^5 t$

(iii) $\cos t \cos 2t \cos 3t$ **(M.U. 1995)** **(iv)** $(\cos ht - \sin ht)^n$

Sol. : (i) First we find the expression for $\sin^5 t$. (Refer to Applied Mathematics I by the same author, Ex. 1 on page 2-27).

Let $x = \cos t + i \sin t \quad \therefore\ \frac{1}{x} = \cos t - i \sin t$

$$\therefore\quad (2i \sin t)^5 = \left(x - \frac{1}{x}\right)^5$$

$$= x^5 - 5x^4 \cdot \frac{1}{x} + 10x^3 \cdot \frac{1}{x^2} - 10x^2 \cdot \frac{1}{x^4} + 5x\frac{1}{x^4} - \frac{1}{x^5}$$

$$\therefore \quad 32\, i^5 \sin^5 t = \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$= 2i \sin 5t - 5\,(2i \sin 3t) + 10\,(2i \sin t)$$

$$\therefore \quad \sin^5 t = \frac{1}{16}(\sin 5t - 5 \sin 3t + 10 \sin t)$$

$$\therefore \quad L(\sin^5 t) = \frac{1}{16}\left[L(\sin 5t) - 5\, L \sin(3t) + 10\, L \sin t\right]$$

$$= \frac{1}{16}\left[\frac{5}{s^2+25} - 5 \cdot \frac{3}{s^2+9} + 10 \cdot \frac{1}{s^2+1}\right] \quad \text{.......... (1)}$$

$$= \frac{5}{16}\left[\frac{1}{s^2+25} - \frac{3}{s^2+9} + \frac{2}{s^2+1}\right]$$

$$= \frac{5}{16}\left[\frac{s^2+9-3s^2-75}{(s^2+25)(s^2+9)} + \frac{2}{s^2+1}\right]$$

$$= \frac{5}{16}\left[\frac{-2s^2-66}{(s^2+25)(s^2+9)} + \frac{2}{s^2+1}\right]$$

$$= \frac{5}{8}\left[\frac{-s^2-33}{(s^2+25)(s^2+9)} + \frac{1}{s^2+1}\right]$$

$$= \frac{5}{8}\left[\frac{-s^4-s^2-33s^2-33+s^4+34s^2+225}{(s^2+1)(s^2+9)(s^2+25)}\right]$$

$$= \frac{5}{8} \cdot \frac{192}{(s^2+1)(s^2+9)(s^2+25)} = \frac{120}{(s^2+1)(s^2+9)(s^2+25)}$$

$$= \frac{5!}{(s^2+1)(s^2+9)(s^2+25)}$$

Aliter : $\sin^5 t = \left(\dfrac{e^{it} - e^{-it}}{2i}\right)^5$

$$= \frac{1}{32\, i}\left[e^{5it} - 5^{4it} \cdot e^{-it} + 10 \cdot e^{3it} \cdot e^{-2it}\right.$$
$$\left. - 10\, e^{2it} \cdot e^{-3it} + 5\, e^{it} \cdot e^{-4it} - e^{-5it}\right]$$

$$= \frac{1}{32i}\left[(e^{5it} - e^{-5it}) - 5\,(e^{3it} - e^{-3it}) + 10\,(e^{it} - e^{-it})\right]$$

$$= \frac{1}{16}\left[\left(\frac{e^{5it} - e^{-5it}}{2i}\right) - 5\left(\frac{e^{3it} - e^{-3it}}{2i}\right) + 10\left(\frac{e^{it} - e^{-it}}{2i}\right)\right]$$

$$= \frac{1}{16}\left[\sin 5t - 5 \sin 3t + 10 \sin t\right]$$

$$\therefore \quad L\left[\sin^5 t\right] = \frac{1}{16}\left[L \sin 5t - 5\, L \sin 3t + 10\, L \sin t\right]$$

$$= \frac{1}{16}\left[\frac{5}{s^2+5^2} - 5 \cdot \frac{3}{s^2+3^2} + 10 \cdot \frac{1}{s^2+1}\right]$$

Same as (1).

(ii) We have, $\sin h^5 t = \left(\dfrac{e^t - e^{-t}}{2}\right)^5$

$$= \frac{1}{32}\left[e^{5t} - 5\, e^{4t} \cdot e^{-t} + 10\, e^{3t} \cdot e^{-2t} - 10\, e^{2t} \cdot e^{-3t} + 5\, e^{t} \cdot e^{-4t} - e^{-5t}\right]$$

$$= \frac{2}{32}\left[\left(\frac{e^{5t} - e^{-5t}}{2}\right) - 5\left(\frac{e^{3t} - e^{-3t}}{2}\right) + 10\left(\frac{e^{t} - e^{-t}}{2}\right)\right]$$

$$= \frac{1}{16}\left[\sin h\, 5t - 5 \sin h\, 3t + 10 \sin h\, t\right]$$

$$\therefore \quad L \sin h^5 t = \frac{1}{16}\left[L \sin h\, 5t - 5 \sin h\, 3t + 10 \sin h\, t\right]$$

$$= \frac{1}{16}\left[\frac{5}{s^2-25} - \frac{15}{s^2-9} + \frac{10}{s^2-1}\right]$$

$$= \frac{5}{16}\left[\frac{1}{s^2-25} - \frac{3}{s^2-9} + \frac{2}{s^2-1}\right]$$

$$= \frac{5}{16}\left[\frac{s^2-9-3s^2+75}{(s^2-25)(s^2-9)} + \frac{2}{s^2-1}\right]$$

$$= \frac{5}{16}\left[\frac{-2s^2+66}{(s^2-25)(s^2-9)} + \frac{2}{s^2-1}\right]$$

$$= \frac{5}{16}\left[\frac{-2s^4+66s^2+2s^2-66+2s^4-68s^2+450}{(s^2-1)(s^2-9)(s^2-25)}\right]$$

$$= \frac{120}{(s^2-1)(s^2-9)(s^2-25)} = \frac{5!}{(s^2-1)(s^2-9)(s^2-25)}$$

(iii) $L(\cos t \cos 2t \cos 3t) = L\left[\dfrac{1}{2}(\cos 3t + \cos t) \cos 3t\right]$

$$= \frac{1}{2} L\left[\cos^2 3t + \cos 3t \cos t\right]$$

$$= \frac{1}{2} L\left[\frac{1}{2}(1+\cos 6t) + \frac{1}{2}\{\cos 4t + \cos 2t\}\right]$$

$$= \frac{1}{4} L\left[1 + \cos 2t + \cos 4t + \cos 6t\right]$$

$$= \frac{1}{4}\left[L(1) + L(\cos 2t) + L(\cos 4t) + L(\cos 6t)\right]$$

$$= \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+2^2} + \frac{s}{s^2+4^2} + \frac{s}{s^2+6^2}\right]$$

(iv) $(\cosh t - \sinh t)^n = \left(\dfrac{e^t + e^{-t}}{2} - \dfrac{e^t - e^{-t}}{2}\right)^n = (e^{-t})^n = e^{-nt}$

$\therefore \quad L[(\cosh t - \sinh t)^n] = L(e^{-nt}) = \dfrac{1}{s+n}$

EXERCISE

Find the Laplace transforms of

1. $(t^2 + a)^2$, **2.** $2t^3 + \cos 4t + e^{-2t}$,

3. $e^{2t} + 4t^3 - \sin 2t \cos 3t$ **4.** $\cos(wt + \beta)$, **(M.U. 1999)**

5. $\cos^2 2bt$, **6.** $e^t + \sin 2t \cdot \sin 3t$, **7.** $3t^2 + e^{-t} + \sin^3 2t$,

8. $\cos^3 2t$, **9.** $\sinh^3 3t$, **10.** $\cosh at - \cos bt$,

11. $\sinh at - \sin bt$, **12.** $\cosh^3 2t$, **13.** $\cos^5 t$, **14.** $\cosh^5 t$,

15. $\cos^4 t$, **16.** $\sin^4 t$ **(M.U. 2002, 03)** **17.** $\cosh^4 t$ **(M.U. 2004)**

18. $(\cosh + \sin ht)^n$ **19.** $\dfrac{1+2t}{\sqrt{2}}$ **20.** $\sqrt{1+\sin t}$ **(M.U. 2004)**

21. $(\sqrt{t} - 1)^2$ **(M.U. 2004)**

[**Ans. : (1)** $\dfrac{a^2 s^4 + 4as^2 + 24}{s^5}$, **(2)** $2 \cdot \dfrac{3!}{s^4} + \dfrac{s}{s^2+4^2} + \dfrac{1}{s+2}$,

(3) $\dfrac{1}{s-2} + 4 \cdot \dfrac{3!}{s^4} - \dfrac{1}{2}\left[\dfrac{5}{s^2+5^2} - \dfrac{1}{s^2+1^2}\right]$,

(4) $\cos\beta \cdot \dfrac{5}{s^2+w^2} - \sin\beta \cdot \dfrac{w}{s^2+w^2}$, **(5)** $\dfrac{1}{2}\left[\dfrac{1}{s} + \dfrac{s}{s^2+16b^2}\right]$,

(6) $\dfrac{1}{s-1} + \dfrac{1}{2}\left[\dfrac{s}{s^2+1^2} - \dfrac{s}{s^2+5^2}\right]$,

(7) $3 \cdot \dfrac{2!}{s^3} + \dfrac{1}{s+1} + \dfrac{3}{4} \cdot \dfrac{2}{s^2+2^2} - \dfrac{1}{4} \cdot \dfrac{6}{s^2+6^2}$, **(8)** $\dfrac{s(s^2+28)}{(s^2+36)(s^2+4)}$,

(9) $\dfrac{162}{(s^2-81)(s^2-8)}$, **(10)** $\dfrac{s}{s^2-a^2} - \dfrac{s}{s^2+b^2}$, **(11)** $\dfrac{a}{s^2-a^2} - \dfrac{b}{s^2+b^2}$,

(12) $\dfrac{s(s^2-28)}{(s^2-36)(s^2-4)}$, **(13)** $\dfrac{s}{16}\left[\dfrac{1}{s^2+25} + \dfrac{5}{s^2+9} + \dfrac{10}{s^2+1}\right]$,

(14) $\dfrac{s}{16}\left[\dfrac{1}{s^2-25} + \dfrac{5}{s^2-9} + \dfrac{10}{s^2-1}\right]$, **(15)** $\dfrac{1}{8}\left[\dfrac{s}{s^2-16} + \dfrac{4s}{s^2-4} + \dfrac{6}{s}\right]$

(16) $\dfrac{1}{4}\left[\dfrac{3s}{2} - \dfrac{2s}{s^2+4} + \dfrac{1}{s^2+16}\right]$ **(17)** $\dfrac{1}{8}\left[\dfrac{3}{s} + \dfrac{4s}{s^2-4} + \dfrac{s}{s^2+16}\right]$

(18) $\dfrac{1}{s-n}$, **(19)** $\sqrt{\dfrac{\pi}{s}}\left(1 + \dfrac{1}{s}\right)$ **(20)** $\dfrac{s}{s^2+(1/2)^2} + \dfrac{1/2}{s^2+(1/2)^2}$

(21) $\dfrac{1}{s^2} - \dfrac{\sqrt{\pi}}{s^{3/2}} + \dfrac{1}{s}$]

Miscellaneous Examples

Ex. 1 : Find the Laplace transform of the following.

(i) $\sin\sqrt{t}$ **(M.U. 1996)** **(ii)** $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$ **(M.U. 2004)**

Sol. : (i) Since, $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots..$

$$\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \ldots..$$

$$\therefore \quad L\left[\sin\sqrt{t}\right] = L(t^{1/2}) - \frac{1}{3!}L(t^{3/2}) + \frac{1}{5!}L(t^{5/2}) + \ldots..$$

But $L(t^n) = \dfrac{\overline{|n+1}}{s^{n+1}}$ and $\overline{|n} = n\overline{|n-1}$, $\overline{|1/2} = \sqrt{\pi}$

$$\therefore \quad L\left[\sin\sqrt{t}\right] = \frac{\overline{|3/2}}{s^{3/2}} - \frac{1}{3!} \cdot \frac{\overline{|5/2}}{s^{5/2}} + \frac{1}{5!} \cdot \frac{\overline{|7/2}}{s^{7/2}} - \ldots..$$

$$= \frac{(1/2)\overline{|1/2}}{s^{3/2}} - \frac{1}{3!} \cdot \frac{(3/2)(1/2)\overline{|1/2}}{s^{5/2}} + \frac{1}{5!} \cdot \frac{(5/2)(3/2)(1/2)\overline{|1/2}}{s^{7/2}}$$

$$= \frac{\overline{|1/2}}{2s^{3/2}}\left[1 - \left(\frac{1}{2^2 \cdot s}\right) + \frac{1}{2!}\left(\frac{1}{2^2 \cdot s}\right)^2 - \ldots..\right] \quad \left[\because \overline{|n} = n\overline{|n-1}\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-1/(4s)} \quad \left[\because \overline{|1/2} = \sqrt{\pi}\right]$$

(ii) We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\therefore \quad \cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \ldots$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \ldots$$

$$\therefore \quad L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\overline{|1/2}}{s^{1/2}} - \frac{1}{2!}\cdot\frac{\overline{|3/2}}{s^{3/2}} + \frac{1}{4!}\cdot\frac{\overline{|5/2}}{s^{5/2}} - \frac{1}{6!}\cdot\frac{\overline{|7/2}}{s^{7/2}} + \ldots$$

$$= \frac{\overline{|1/2}}{s^{1/2}} - \frac{1}{2!}\cdot\frac{(1/2)\overline{|1/2}}{s^{3/2}} + \frac{1}{4!}\cdot\frac{(3/2)(1/2)\overline{|1/2}}{s^{5/2}} - \frac{1}{6!}\cdot\frac{(5/2)(3/2)(1/2)\overline{|1/2}}{s^{7/2}} + \ldots$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \frac{1}{3!(4s)^3} + \ldots\right] = \sqrt{\frac{\pi}{s}}\cdot e^{-1/4s}.$$

Ex. 2 : If $J_0(t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2}\left(\frac{t}{2}\right)^{2r}$, find $L[J_0(t)]$.

Sol. : By data $J_0(t) = 1 - \left(\frac{t}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{t}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{t}{2}\right)^6 + \ldots$

$$= 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \frac{t^6}{2^2\cdot 4^2\cdot 6^2} + \ldots$$

$$\therefore \quad L[J_0(t)] = L(1) - \frac{1}{2^2}L(t^2) + \frac{1}{2^2\cdot 4^2}\cdot L(t^4) - \frac{1}{2^2\cdot 4^2\cdot 6^2}\cdot L(t^6) - \ldots$$

$$= \frac{1}{s} - \frac{1}{2^2}\cdot\frac{2!}{s^3} + \frac{1}{2^2\cdot 4^2}\cdot\frac{4!}{s^5} - \frac{1}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{6!}{s^7} + \ldots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{2}\cdot\frac{3}{4}\left(\frac{1}{s^2}\right)^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\left(\frac{1}{s^2}\right)^3 + \ldots\right]$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{1+s^2}}.$$

Ex. 3 : Find $L(erf\sqrt{t})$. **(M.U. 1996, 97, 2003)**

Sol. : We know that $erf\, t = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx$

$$\therefore \quad erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx$$

$$\therefore \quad erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right]dx$$

$$= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \ldots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5(2!)} - \frac{t^{7/2}}{7(3!)} + \ldots\right]$$

Taking Laplace transform of both sides and using $L(t^n) = \frac{\overline{|n+1}}{s^{n+1}}$

$$L(erf\sqrt{t}) = \frac{2}{\sqrt{\pi}}\left[\frac{\overline{|3/2}}{s^{3/2}} - \frac{1}{3}\frac{\overline{|5/2}}{s^{5/2}} + \frac{1}{5(2!)}\frac{\overline{|7/2}}{s^{7/2}} + \ldots\right]$$

But $\overline{|3/2} = (1/2)\overline{|1/2} = (1/2)\sqrt{\pi}$

$\overline{|5/2} = (3/2)(\overline{|3/2}) = (3/2)(1/2)\sqrt{\pi}$ etc.

$$\therefore \quad L(erf\sqrt{t}) = \frac{1}{s^{3/2}}\left[1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{s^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{s^3} + \ldots\right]$$

$$= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s^{3/2}}\frac{\sqrt{s}}{\sqrt{s+1}} = \frac{1}{s\sqrt{s+1}}$$

(We shall obtain this result again on page 3-53.)

Ex. 4 : If $J_0(t) = \frac{1}{\pi}\int_0^{\pi}\cos(t\cos\theta)\,d\theta$, prove that $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$.

Hence, evaluate $\frac{1}{\pi}\int_0^{\infty} e^{-t}\left[\int_0^{\pi}\cos t\cos\theta\; d\theta\right]dt$. **(M.U. 2003)**

Sol. : We have $J_0(t) = \frac{1}{\pi}\int_0^{\pi}\cos(t\cos\theta)\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta$

Taking Laplace transforms of both sides,

$$L[J_0(t)] = \frac{2}{\pi}\int_0^{\infty} e^{-st}\left[\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta\right]dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left[\int_0^{\infty} e^{-st}\cos(t\cos\theta)\,d\theta\right]dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2}[L\cos(t\cos\theta)]\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{s}{s^2+\cos^2\theta}\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\frac{\sec^2\theta}{s^2\sec^2\theta+1}\,d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\frac{\sec^2\theta}{(s^2+1)+s^2\tan^2\theta}\,d\theta$$

Put $s\tan\theta = t \quad \therefore \; s\sec^2\theta\, d\theta = dt$

When $\theta = 0$, $t = 0$; when $\theta = \pi/2$, $t = \infty$.

$$\therefore \quad L[J_o(t)] = \frac{2}{\pi}\int_0^{\infty}\frac{dt}{t^2+(s^2+1)}$$

$$= \frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\cdot\left[\tan^{-1}\left(\frac{t}{\sqrt{s^2+1}}\right)\right]_0^{\infty}$$

$$= \frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\left[\frac{\pi}{2}-0\right] = \frac{1}{\sqrt{s^2+1}}$$

By definition of Laplace transform this means

$$\int_0^{\infty} e^{-st}\cdot J_o(t)\,dt = \frac{1}{\sqrt{s^2+1}}$$

i.e. $$\int_0^{\infty} e^{-st}\cdot\left[\frac{1}{\pi}\int_0^{\pi}\cos(t\cos\theta)\,d\theta\right]dt = \frac{1}{\sqrt{s^2+1}}$$

Putting $s = 1$, we get

$$\frac{1}{\pi}\int_0^{\infty} e^{-t}\left[\int_0^{\infty}\cos(t\cos\theta)\,d\theta\right]dt = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$$

Ex. 5 : If $J_o(t) = \frac{1}{\pi}\int_0^{\pi}\cos(t\sin\theta)\,d\theta$, prove that $L[J_o(t)] = \frac{1}{\sqrt{s^2+1}}$.

Sol. : Prove it yourself on the above lines.

(a) Particular Value of the Function $L[f(t)]$: $L[f(t)]$ or $\Phi(s)$ is a function of s and for different values of s we get different values of $L[f(t)]$ or $\Phi(s)$.

For example, by definition,

$$\int_0^{\infty} e^{-st}\sin t\,dt = \frac{1}{s^2+1^2} \qquad \therefore \int_0^{\infty} e^{-3t}\sin t\,dt = \frac{1}{10} \text{ for } s = 3.$$

Also $$\int_0^{\infty} e^{-st}\cos 3t\,dt = \frac{s}{s^2+9} \qquad \therefore \int_0^{\infty} e^{-2t}\cos 3t\,dt = \frac{2}{13} \text{ for } s = 2.$$

Ex. 1 : Evaluate $\int_0^{\infty} e^{-2t}\cdot\sin^3 t\,dt$. **(M.U. 1985, 2002)**

Sol. : $$\int_0^{\infty} e^{-st}\cdot\sin^3 t\cdot dt = L(\sin^3 t) = L\left[\frac{3}{4}\cdot(\sin t) - \frac{1}{4}\cdot(\sin 3t)\right]$$

$$= \frac{3}{4}\cdot\frac{1}{s^2+1} - \frac{1}{4}\cdot\frac{3}{s^2+9} \qquad \text{Putting } s = 2,$$

$$\int_0^{\infty} e^{-2t}\cdot\sin^3 t\cdot dt = \frac{3}{4}\cdot\frac{1}{4+1} - \frac{1}{4}\cdot\frac{3}{4+9} = \frac{6}{65}$$

Ex. 2 : Evaluate $\int_0^{\infty} e^{-2t}\cos h^5 t\,dt$.

Sol. : We have $\cos h^5 x = \left(\frac{e^x+e^{-x}}{2}\right)^5$

$$= \frac{1}{32}\left[e^{5x} + 5e^{4x}\cdot e^{-x} + 10e^{3x}\cdot e^{-2x} + 10e^{2x}\cdot e^{-3x} + 5e^x\cdot e^{-4x} + e^{-5x}\right]$$

$$= \frac{2}{32}\left[\left(\frac{e^{5x}+e^{-5x}}{2}\right) + 5\left(\frac{e^{3x}+e^{-3x}}{2}\right) + 10\left(\frac{e^x+e^{-x}}{2}\right)\right]$$

$$= \frac{1}{16}[\cos h\,5x + 5\cos h\,3x + 10\cos h\,x]$$

$$\therefore \int_0^{\infty} e^{-st}\cos h^5 t\,dt = \frac{1}{16}[L(\cos h\,5t) + 5L(\cos h\,3t) + 10\,L(\cos ht)]$$

$$= \frac{1}{16}\left[\frac{s}{s^2-25} + 5\cdot\frac{s}{s^2-9} + 10\cdot\frac{s}{s^2-1}\right]$$

Now put $s = 2$.

$$\therefore \int_0^{\infty} e^{-2t}\cos h^5 t\,dt = \frac{1}{16}\left[\frac{2}{4-25} + \frac{5\cdot 2}{4-9} + \frac{10\cdot 2}{4-1}\right]$$

$$= \frac{1}{16}\left[-\frac{2}{21} - \frac{10}{5} + \frac{20}{3}\right] = \frac{1}{16}\cdot\frac{32}{7} = \frac{2}{7}.$$

Ex. 3 : Find $L(erf\sqrt{t})$ and hence, obtain $\int_0^{\infty} erf\sqrt{t}\,e^{-t}\,dt$. **(M.U. 2000)**

Sol. : We have proved (page 3-12) above that $L(erf\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$.

$$\therefore \int_0^{\infty} e^{-st} erf\sqrt{t}\,dt = \frac{1}{s\sqrt{s+1}}$$

Putting $s = 1$,

$$\int_0^{\infty} e^{-t} erf\sqrt{t}\,dt = \frac{1}{1\sqrt{1+1}} = \frac{1}{\sqrt{2}}.$$

EXERCISE

(A) Evaluate by Laplace transform,

1. $\int_0^\infty e^{-3t} \cdot \cos^2 t \, dt$, 2. $\int_0^\infty e^{-4t} \cdot \cos h^3 t \, dt$,

3. $\int_0^\infty e^{-5t} \cdot \sin h^3 \, dt$, 4. $\int_0^\infty e^{-3t} \, t^5 \, dt$, 5. $\int_0^\infty e^{-4t} \cdot \sin^3 t \, dt$,

6. $\int_0^\infty e^{-3t} \cdot \cos^3 t \, dt$. **(M.U. 2003)**

(B) If $\int_0^\infty e^{-2t} \sin(t+\alpha)\cos(t-\alpha)\, dt = \frac{3}{8}$, then find α. **(M.U. 2004)**

[Ans : (A) (1) $\frac{11}{39}$, **(2)** $\frac{12}{35}$, **(3)** $\frac{1}{64}$, **(4)** $\frac{40}{243}$, **(5)** $\frac{6}{425}$, **(6)** $\frac{4}{15}$. **(B)** $\alpha = \frac{\pi}{4}$ **]**

5. Change Of Scale Property

If $L[f(t)] = \Phi(s)$, then $\boxed{L[f(at)] = \frac{1}{a}\Phi\left(\frac{s}{a}\right)}$ .... (11)

(M.U. 2002)

Proof : We have $L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt$. Now, put $u = at$.

$$= \int_0^\infty e^{-s(u/a)} f(u) \cdot \frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-(s/a)u} f(u)\, du = \frac{1}{a}\Phi\left(\frac{s}{a}\right).$$

e.g. (i) If $L f(t) = \frac{2s}{s^2+4}$, then $L[f(2t)] = \frac{1}{2}\left[\frac{2(s/2)}{(s/2)^2+4}\right] = \frac{2s}{s^2+16}$.

(ii) If $L f(t) = \frac{s^2-s}{s^2+3s+5}$, then

$$L[f(3t)] = \frac{1}{3}\left[\frac{(s/3)^2-(s/3)}{(s/3)^2+3(s/3)+5}\right] = \frac{1}{3}\left[\frac{s^2-3s}{s^2+9s+45}\right]$$

Ex. 1 : (i) If $L f(t) = \frac{1}{s}e^{-1/s}$, find $L f(3t)$.

(ii) If $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$, find $L\left(\sqrt{\frac{\pi}{t}}\right)$.

(iii) If $L(erf\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$, find $L(erf\, 3\sqrt{t})$.

(iv) If $L J_o(t) = \frac{1}{\sqrt{s^2+1}}$, find $L J_o(at)$.

Sol. : We have if $L f(t) = \Phi(s)$, then $L f(at) = \frac{1}{a}\Phi\left(\frac{s}{a}\right)$.

(i) Since, $L f(t) = \frac{1}{s}e^{-1/s}$, $L f(3t) = \frac{1}{3}\cdot\frac{1}{(s/3)}\cdot e^{-1/(s/3)}$

$$\therefore \; L f(3t) = \frac{1}{s}\cdot e^{-3/s}.$$

(ii) Since, $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$, $L\left(\sqrt{\frac{\pi}{t}}\right) = L\left(\pi\sqrt{\frac{1}{\pi t}}\right)$

$$\therefore \; L\left(\sqrt{\frac{\pi}{t}}\right) = \frac{1}{\pi\sqrt{s/\pi}} = \sqrt{\frac{\pi}{s}}$$

(iii) Since, $L(erf\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$

$$\therefore \; L(erf\; 3\sqrt{t}) = L(erf\sqrt{9t})$$

$$\therefore \; L(erf\, 3\sqrt{t}) = \frac{1}{3}\cdot\frac{1}{(s/3)\sqrt{(s/9)+1}} = \frac{3}{s\sqrt{s+9}}$$

(iv) Since, $L J_o(t) = \frac{1}{\sqrt{s^2+1}}$,

$$\therefore \; L J_o(at) = \frac{1}{a}\cdot\frac{1}{\sqrt{(s^2/a^2)+1}} = \frac{1}{\sqrt{s^2+a^2}}.$$

Ex. 2 : Find $L\{erf\sqrt{t}\}$ and then evaluate $\int_0^\infty e^{-2t}\, erf(2\sqrt{t})\, dt$.

(M.U. 2003)

Sol. : We have proved (Ex. 3, page 3-12) that $L\{erf\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}$.

But $L\{f(at)\} = \frac{1}{a}\Phi\left(\frac{s}{a}\right)$

$$\therefore \; L\{erf(2\sqrt{t})\} = L\{erf.(\sqrt{4t})\}$$

$$= \frac{1}{4}\cdot\frac{1}{(s/4)\sqrt{(s/4)+1}} = \frac{2}{s\sqrt{s+4}}$$

$$\therefore \; \int_0^\infty e^{-st}\cdot erf(2\sqrt{t})\, dt = \frac{2}{s\sqrt{s+4}}$$

Putting $s = 2$,

$$\int_0^\infty e^{-2t} erf(2\sqrt{t})\, dt = \frac{2}{2\sqrt{2+4}} = \frac{1}{\sqrt{6}}$$

EXERICSE

1. If $Lf(t)=\frac{2}{s^3}e^{-s}$, find $Lf(2t)$. [Ans.: $\frac{8}{s^{-3}}e^{-s/2}$]

2. If $L\,erf_c(\sqrt{t})=\frac{1}{\sqrt{s+1}+1}$, find $L\,erf_c(2\sqrt{t})$. [Ans.: $\frac{1}{2\sqrt{s+4}+4}$]

3. If $Lf(t)=\log\left(\frac{s+3}{s+1}\right)$, find $Lf(2t)$. [Ans.: $\log\sqrt{\frac{s+6}{s+2}}$]

4. If $L\left[\sin\sqrt{t}\right]=\frac{\sqrt{\pi}}{2s\sqrt{s}}\cdot e^{-1/4s}$, find $L\left[\sin 2\sqrt{t}\right]$. (M.U. 2004)

[Ans.: $\frac{\sqrt{\pi}}{s\sqrt{s}}\cdot e^{-1/s}$]

6. First Shifting Theorem

If $L[f(t)]=F(s)$, then $\boxed{L[e^{-at}f(t)]=\Phi(s+a)}$ .... (12)

(M.U. 1997, 2003)

Proof : We have $L[e^{-at}f(t)]=\int_0^{\infty}e^{-st}\left\{e^{-at}f(t)\right\}dt$

$$=\int_0^{\infty}e^{-(s+a)t}f(t)\,dt=\Phi(s+a)$$

e.g. $L[\sin at]=\frac{a}{s^2+a^2}$ $\therefore L\left[e^{-bt}\sin at\right]=\frac{a}{(s+b)^2+a^2}$

Cor. : Changing sign of a, we get,

If $L[f(t)]=\Phi(s)$, then $\boxed{L\left[e^{at}f(t)\right]=\Phi(s-a)}$

1. $\because L(\sin at)=\frac{a}{s^2+a^2}$, $L(e^{bt}\sin at)=\frac{a}{(s-b)^2+a^2}$

2. $\because L(\cos at)=\frac{s}{s^2+a^2}$, $L(e^{bt}\cos at)=\frac{s-b}{(s-b)^2+a^2}$

3. $\because L(\sin h\,at)=\frac{a}{s^2-a^2}$, $L(e^{bt}\sin h\,at)=\frac{a}{(s-b)^2-a^2}$

4. $\because L(\cos h\,at)=\frac{s}{s^2-a^2}$, $L(e^{bt}\cos h\,at)=\frac{s-b}{(s-b)^2-a^2}$

5. $\because L(t^n)=\frac{n!}{s^{n+1}}$, $L(e^{bt}\cdot t^n)=\frac{n!}{(s-b)^{n+1}}$ (n is an integer)

6. $\because L(t^n)=\frac{\overline{|n+1}}{s^{n+1}}$, $L(e^{bt}\cdot t^n)=\frac{\overline{|n+1}}{(s-b)^{n+1}}$

EXERCISE

Write down the Laplace transforms of the following.

1. $e^{3t}\sin 2t$, 2. $e^{-2t}\cos 3t$, 3. $e^{-t}\sin 4t$, 4. $e^{t}\cos t$,
5. $e^{2t}\sin h\,t$, 6. $e^{-t}\cos h\,2t$, 7. $e^{-2t}\sin h\,4t$, 8. $e^{-3t}\cos h\,3t$,
9. $e^{-3t}t^4$, 10. $e^{2t}t^{3/2}$, 11. $e^{-t}\sqrt{t}$, 12. $t\cos 3t$ (M.U. 2003)

[**Ans. :** Answers not given for obvious reason.]

Ex. 1 : Find the Laplace transform of

(i) $e^{4t}\sin^3 t$, (ii) $\cos h\,2t\cos 2t$

Sol. : (i) $L[\sin^3 t]=L\left[\frac{1}{4}(3\sin t-\sin 3t)\right]=\frac{3}{4}L(\sin t)-\frac{1}{4}L(\sin 3t)$

$$=\frac{3}{4}\cdot\frac{1}{s^2+1}-\frac{1}{4}\cdot\frac{3}{s^2+9}.$$

By shifting theorem,

$$L\left[e^{4t}\sin^3 t\right]=\frac{3}{4}\cdot\frac{1}{(s-4)^2+1}-\frac{1}{4}\cdot\frac{3}{(s-4)^2+9}$$

$$=\frac{3}{4}\left[\frac{1}{s^2-8s+17}-\frac{1}{s^2-8s+25}\right]$$

$$=\frac{6}{(s^2-8s+17)(s^2-8s+25)}.$$

(ii) $L[\cos h\,2t\cdot\cos 2t]=L\left[\frac{1}{2}\left(e^{2t}+e^{-2t}\right)\cos 2t\right]$

$$=\frac{1}{2}\left[L(e^{2t}\cos 2t)+L(e^{-2t}\cos 2t)\right]$$

But $L\cos 2t=\frac{s}{s^2+4}$. $\therefore$ By shifting theorem,

$$L\left[\cos h\,2t\cdot\cos 2t\right]=\frac{1}{2}\left[\frac{s-2}{(s-2)^2+4}+\frac{s+2}{(s+2)^2+4}\right]=\frac{s^3}{s^4+64}.$$

Ex. 2 : Find Laplace transform of $\sin h\,at\sin at$. (M.U. 2003, 05)

Sol. : We have $\sin h\,at\sin at=\left(\frac{e^{at}-e^{-at}}{2}\right)\sin at$

$$=\frac{1}{2}\left[e^{at}\sin at-e^{-at}\sin at\right]$$

Now, $L(\sin at)=\frac{a}{s^2+a^2}$.

By first shifting theorem (12, page 3-18)

$$L(e^{at}\sin at) = \frac{a}{(s-a)^2+a^2} = \frac{a}{s^2-2as+2a^2}$$

$$L(e^{-at}\sin at) = \frac{a}{(s+a)^2+a^2} = \frac{a}{s^2+2as+2a^2}$$

$$\therefore\ L(\sin h\, at \sin at) = \frac{1}{2}\left[L(e^{at}\sin at) - L(e^{-at}\sin at)\right]$$

$$= \frac{1}{2}\cdot\left[\frac{a}{s^2-2as+2a^2} - \frac{a}{s^2+2as+2a^2}\right]$$

$$= \frac{a}{2}\cdot\left[\frac{4as}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{2a^2 s}{(s^2+2a^2-2as)(s^2+2a^2+2as)} = \frac{2a^2 s}{s^4+4a^4}$$

Ex. 3 : Show that $L[\sin h(t/2)\sin(t/2)] = \dfrac{2s}{4s^4+1}$.

Sol. : Putting $a = 1/2$ in the above example, we get

$$L[\sin h(t/2)\sin(t/2)] = \frac{2(1/4)s}{s^4+4\cdot(1/2)^4}$$

$$= \frac{2s}{4s^4+1}$$

Or solve it independently.

Ex. 4 : Show that

$$L\left[\sin h(t/2)\sin(\sqrt{3}\,t/2)\right] = \frac{\sqrt{3}}{2}\cdot\frac{s}{(s^4+s^2+1)}.$$

(M.U. 1993, 2002, 03)

Sol. : We have, $\sin h(t/2)\cdot\sin(\sqrt{3}t/2) = \left(\dfrac{e^{t/2}-e^{-t/2}}{2}\right)\cdot\sin\dfrac{\sqrt{3}}{2}t$

Now $L\sin\left(\dfrac{\sqrt{3}}{2}t\right) = \dfrac{\sqrt{3}/2}{s^2+(3/4)}$.

By first shifting theorem,

$$\therefore\ L e^{t/2}\cdot\sin\left(\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{[s-(1/2)]^2+3/4} = \frac{\sqrt{3}/2}{s^2+1-s}$$

$$L e^{-t/2}\cdot\sin\left(\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{[s+(1/2)]^2+3/4} = \frac{\sqrt{3}/2}{s^2+1+s}$$

$$\therefore\ L\left[\sin h(t/2)\sin(\sqrt{3}t/2)\right] = \frac{1}{2}\left[\frac{\sqrt{3}/2}{(s^2+1)-s} - \frac{\sqrt{3}/2}{(s^2+1)+s}\right]$$

$$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\left[\frac{(s^2+1+s)-(s^2+1-s)}{(s^2+1)^2-s^2}\right]$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{s}{s^4+s^2+1}$$

Ex. 5 : Find the Laplace transform of the following.

(i) $\dfrac{\cos 2t \sin t}{e^t}$, **(M.U. 1993, 2003, 04)** (ii) $e^{-4t}\sin h\, t\sin t$, **(M.U. 1995)**

(iii) $e^t \sin 2t \sin 3t$, **(M.U. 1997)** (iv) $e^{-3t}\cos h\, 5t \sin 4t$. **(M.U. 1998)**

Sol. : (i) $\cos 2t \sin t = \dfrac{1}{2}\cdot 2\sin t\cos 2t = \dfrac{1}{2}[\sin(1+2)t + \sin(1-2)t]$

$$= \frac{1}{2}[\sin 3t - \sin t]$$

$$\therefore\ L(\cos 2t\sin t) = \frac{1}{2}[L(\sin 3t) - L(\sin t)] = \frac{1}{2}\cdot\left[\frac{3}{s^2+9} - \frac{1}{s^2+1}\right]$$

Now by shifting theorem,

$$L\left[e^{-t}(\cos 2t\sin t)\right] = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+2s+10} - \frac{1}{s^2+2s+2}\right]$$

$$= \frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$$

(ii) $e^{-4t}\sin ht\sin t = e^{-4t}\cdot\left(\dfrac{e^t-e^{-t}}{2}\right)\sin t$

$$= \frac{1}{2}e^{-3t}\sin t - \frac{1}{2}e^{-5t}\sin t$$

$$\therefore\ L\left(e^{-4t}\sin ht\sin t\right) = \frac{1}{2}L\left(e^{-3t}\sin t\right) - \frac{1}{2}L\left(e^{-5t}\sin t\right)$$

$$= \frac{1}{2}\cdot\frac{1}{(s+3)^2+1^2} - \frac{1}{2}\cdot\frac{1}{(s+5)^2+1^2}$$

$$= \frac{1}{2}\left[\frac{s^2+10s+26-s^2-6s-10}{(s^2+6s+10)(s^2+10s+26)}\right]$$

$$= \frac{2(s+4)}{(s^2+6s+10)(s^2+10s+26)}.$$

(iii) $\sin 2t \sin 3t = \frac{1}{2} \cdot 2 \sin 3t \sin 2t = -\frac{1}{2}[\cos 5t - \cos t]$

$$\therefore \quad L(\sin 2t \sin 3t) = -\frac{1}{2}[L(\cos 5t) - L(\cos t)] = -\frac{1}{2}\left[\frac{s}{s^2+5^2} - \frac{s}{s^2+1^2}\right]$$

Now by shifting theorem,

$$L\left[e^t \sin 2t \sin 3t\right] = -\frac{1}{2}\left[\frac{s-1}{(s-1)^2+5^2} - \frac{s-1}{(s-1)^2+1^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-1}{s^2-2s+2} - \frac{s-1}{s^2-2s+26}\right]$$

$$= \frac{12(s-1)}{(s^2-2s+2)(s^2-2s+26)}$$

(iv) We have $\quad e^{-3t} \cos h\, 5t \sin 4t$

$$= e^{-3t}\left(\frac{e^{5t}+e^{-5t}}{2}\right)\sin 4t = \frac{1}{2}\left(e^{2t}+e^{-8t}\right)\sin 4t$$

Now $L(\sin 4t) = \frac{4}{s^2+16}$

$\therefore$ By first shifting theorem,

$$L\left[e^{-3t} \cos h\, 5t \sin 4t\right] = \frac{1}{2} L\left[(e^{2t}+e^{-8t})\sin 4t\right]$$

$$= \frac{1}{2}\left[\frac{4}{(s-2)^2+16} + \frac{4}{(s+8)^2+16}\right]$$

$$= \frac{4(s^2+6s+50)}{(s^2-4s+20)(s^2+16s+80)}$$

Ex. 6 : Find Laplace transform of $\sin 2t \cos t \cos h\, 2t$. **(M.U. 1994)**

Sol. : $\sin 2t \cos t = \frac{1}{2} \cdot 2 \sin 2t \cos t = \frac{1}{2}[\sin 3t + \sin t]$

$$\cos h\, 2t = \frac{e^{2t}+e^{-2t}}{2}$$

$$\therefore \quad \sin 2t \cos t \cos h\, 2t = \frac{1}{2}(e^{2t}+e^{-2t})(\sin 3t + \sin t) \qquad \text{.......... (1)}$$

$$\therefore \quad L \sin 3t = \frac{3}{s^2+9}$$

$$\therefore \quad L\left[e^{2t}\sin 3t\right] = \frac{3}{(s-2)^2+9}, \quad L\left[e^{-2t}\sin 3t\right] = \frac{3}{(s+2)^2+9}$$

$$\therefore \quad L\left(e^{2t}\sin 3t\right) + L\left(e^{-2t}\sin 3t\right) = 3\left[\frac{1}{(s-2)^2+9} + \frac{1}{(s+2)^2+9}\right]$$

$$= \frac{3 \cdot 2(s^2+13)}{s^4+10s^2+13^2} \qquad \text{.......... (2)}$$

Now $\sin t = \frac{1}{s^2+1}$

$$\therefore \quad L(e^{2t}\sin t) = \frac{1}{(s-2)^2+1}, \quad L(e^{-2t}\sin t) = \frac{1}{(s+2)^2+1}$$

$$\therefore \quad L(e^{2t}\sin t) + L(e^{-2t}\sin t) = \frac{2(s^2+5)}{s^4-6s^2+5^2} \qquad \text{.......... (3)}$$

From (1), (2) and (3), we get

$$L\left[\sin 2t \cos t \cos h\, 2t\right] = \frac{3(s^2+13)}{s^4+10s^2+13^2} + \frac{s^2+5}{s^4-6s^2+5^2}$$

EXERCISE

(A) Find the Laplace transform of

1. $\cos at \cdot \sin h\, at$
2. $e^t \sin 4t \cdot \cos 2t$
3. $e^{-2t}(2\cos 3t - 3\sin 3t)$,
4. $\cos h\, at \sin at$
5. $\cos 3t \cdot \cos h\, 4t$
6. $t^n e^{-at} + \sin^3 t$
7. $\cos h\, at \cos at$
8. $t^2 - 3t + 5 + e^{2t}\, t^2$
9. $\cos h\, at \cos bt$
10. $e^{4t}\, t^{3/2}$
11. $e^{-4t} \cos ht \sin t$ **(M.U. 1995)**
12. $e^{-3t} \cos h\, 4t \sin 3t$ **(M.U. 1999)**
13. $e^{2t} \cos 2t \cos t$ **(M.U. 2002, 03)**
14. $e^{2t}(1+t)^2$ **(M.U. 2002, 03)**

[**Ans. : (1)** $\frac{a(s^2-2a^2)}{s^4+4a^4}$, **(2)** $\frac{3}{(s-1)^2+6^2} + \frac{1}{(s-1)^2+2^2}$,

(3) $\frac{2s-5}{s^2+4s+13}$, **(4)** $\frac{a(s^2+2a^2)}{s^4+2a^4}$, **(5)** $\frac{s(s^2-7)}{s^4+25^2-14s^2}$,

(6) $\frac{\lfloor n+1}{(s+a)^{n+1}} + \frac{6}{(s^2+1)(s^2+9)}$, **(7)** $\frac{s^3}{s^4+4a^4}$,

(8) $\frac{5s^2-3s+2}{s^3} + \frac{2}{(s-2)^3}$, **(9)** $\frac{s[s^2+(b^2-a^2)]}{s^4+2(b^2-a^2)s^2+(a^2+b^2)^2}$,

(10) $\dfrac{\sqrt{5/2}}{(s-4)^{5/2}}$, (11) $\dfrac{s^2+8s+18}{(s^2+6s+10)(s^2+10s+26)}$,

(12) $\dfrac{3(s^2+6s+34)}{(s^2-2s+10)(s^2+14s+58)}$, (13) $\dfrac{(s-2)(s^2-4s+9)}{(s^2-4s+13)(s^2-4s+5)}$

(14) $\dfrac{1}{s-2}+\dfrac{2}{(s-2)^2}+\dfrac{2}{(s-2)^3}$.]

(B) If $L[f(t)]=\dfrac{s}{s^2+s+4}$, find $L[e^{-3t}f(2t)]$.

(M.U. 2003) [Ans. : $\dfrac{s+3}{s^2+8s+10}$]

7. Second Shifting Theorem

If $L[f(t)]=\Phi(s)$ and $g(t)=f(t-a)$ when $t>a$ and $g(t)=0$ when $t<a$, then prove that

$$\boxed{L[g(t)]=e^{-as}\Phi(s)} \quad \text{(M.U. 1995, 2002)} \qquad \text{.......... (13)}$$

Proof : By definition of Laplace transform,

$$L\{g(t)\}=\int_0^\infty e^{-st}g(t)\,dt$$

$$=\int_0^a e^{-st}g(t)\,dt+\int_a^\infty e^{-st}g(t)\,dt$$

$$=\int_0^a e^{-st}\cdot 0\,dt+\int_a^\infty e^{-st}f(t-a)\,dt$$

Now put $t-a=p \quad \therefore \quad dt=dp$

$$\therefore \quad L\{g(t)\}=\int_0^\infty e^{-s(a+p)}f(p)\,dp$$

$$=e^{-as}\int_0^\infty e^{-sp}f(p)\,dp=e^{-as}\int_0^\infty e^{-st}f(t)\,dt$$

$$=e^{-as}L[f(t)]=e^{-as}\Phi(s)$$

Ex. : Using second shifting theorem find,

(i) $L[f(t)]$ where $f(t)=\cos(t-\alpha)$, $t>\alpha$ and $f(t)=0$, $t<\alpha$.

(ii) $L[f(t)]$ where $f(t)=e^{t-k}$, $t>k$ and $f(t)=0$, $t<k$.

(iii) $L[f(t)]$ where $f(t)=\sin(t-\pi/3)$, $t>\pi/3$ and $f(t)=0$, $t<\pi/3$.

Sol. : (i) We have $L\cos t=\dfrac{s}{s^2+1}$. Hence, by second shifting theorem,

$$L\cos(t-\alpha)=e^{-\alpha s}\cdot\frac{s}{s^2+1}.$$

Note : We have obtained this result in Ex. 2(2) on page 3-2.

(ii) We have $L(e^t)=\dfrac{1}{s-1}$. Hence, by second shifting theorem,

$$L(e^{t-k})=e^{-ks}\frac{1}{s-1}.$$

(iii) We have $L\sin t=\dfrac{1}{s^2+1}$. Hence, by second shifting theorem

$$L\sin(t-\pi/3)=e^{-\pi s/3}\cdot\frac{1}{s^2+1}.$$

(**Remark :** Solve the above examples (ii) and (iii) by using the definition of $Lf(t)$).

EXERCISE

Using second shifting theorem find $L\,f(t)$ where,

1. $f(t)=\cos[t-(2\pi/3)]$, $t>2\pi/3$ and $f(t)=0$, $t<2\pi/3$.
2. $f(t)=\sin[t-(2\pi/3)]$, $t>2\pi/3$ and $f(t)=0$, $t<2\pi/3$.
3. $f(t)=(t-1)^3$, $t>1$ and $f(t)=0$, $t<1$.
4. $f(t)=(t-2)^2$, $t>2$ and $f(t)=0$, $t<2$.
5. $f(t)=(t-1)^4$, $t>1$ and $f(t)=0$, $t<1$.

[**Ans. :** (1) $e^{-2\pi s/3}\cdot\dfrac{s}{s^2+1}$ (2) $e^{-2\pi s/3}\cdot\dfrac{1}{s^2+1}$ (3) $e^{-s}\cdot\dfrac{3!}{s^4}$ (4) $e^{-2s}\cdot\dfrac{2}{s^3}$ (5) $e^{-s}\cdot\dfrac{24}{s^5}$]

8. Effect Of Multiplication By t

If $L[f(t)]=\Phi(s)$, then $\boxed{L[t^n f(t)]=(-1)^n\dfrac{d^n}{ds^n}\Phi(s)}$ .... (14)

Proof : We shall prove this property by the method of induction.

Step 1 : Let $\Phi(s)=L[f(t)]=\int_0^\infty e^{-st}f(t)\,dt$

Differentiating both sides w.r.t. s and applying the rule of differentiation under integral sign

$$\Phi'(s)=\int_0^\infty\frac{\partial}{\partial s}\left[e^{-st}f(t)\,dt\right]=-\int_0^\infty e^{-st}\,t\,f(t)\,dt=-L[t\,f(t)]$$

$$\therefore \quad L[t\,f(t)]=(-1)\frac{d}{ds}\Phi(s) \qquad \text{.......... (1)}$$

Thus, the rule is true for $n=1$.

Step 2 : Now we assume the rule is true for $n = m$ and prove that it is true for $n = m+1$. *i.e.* we assume that

$$L[t^m f(t)] = (-1)^m \frac{d^m}{ds^m}\Phi(s)$$

$$\therefore \quad (-1)^m \frac{d^m}{ds^m}\Phi(s) = L[t^m f(t)] = \int_0^\infty e^{-st}\, t^m f(t)\, dt$$

Differentiating both sides w.r.t. s and applying the rule of differentiation under the integral sign.

$$(-1)^m \frac{d^{m+1}}{ds^{m+1}}\Phi(s) = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st}\cdot t^m f(t)\, dt\right]$$

$$= -\int_0^\infty e^{-st}\cdot t^{m+1} f(t)\, dt = -L[t^{m+1} f(t)]$$

$$\therefore \quad L\left[t^{m+1} f(t)\right] = (-1)^{m+1}\frac{d^{m+1}}{ds^{m+1}}\Phi(s)$$

Thus, if the property is true form $n = m$ then it is true for $n = m+1$.

Step 3 : Since, the property is true for $n = 1$, it is by step (3) true for $n = 1 + 1 = 2$. Since, it is true for $n = 2$, by step (3) again it is true for $n = 2 + 1 = 3$. Hence, it is true for any value of n.

Note : In particular

If $L\, f(t) = \Phi(s)$, then $\boxed{L[t\, f(t)] = -\Phi'(s), \quad L[t^2 f(t)] = \Phi''(s)}$

Ex. 1 : If $L[erf\sqrt{t}] = \dfrac{1}{s\sqrt{s+1}}$, find $L[t\, erf\, 2\sqrt{t}]$. **(M.U. 2003)**

Sol. : By change of scale property if $L[f(t)] = \Phi(s)$, then $L[f(at)] = \dfrac{1}{a}\Phi\left(\dfrac{s}{a}\right)$.

Since, $L\left[erf\sqrt{t}\right] = \dfrac{1}{s\sqrt{s+1}}$,

$$L\left[erf\, 2\sqrt{t}\right] = L\left[erf\sqrt{4t}\right] = \frac{1}{4}\,\frac{1}{(s/4)\sqrt{(s/4)+1}} = \frac{2}{s\sqrt{s+4}} = \Phi(s)$$

(We have found $L\left[erf\sqrt{t}\right]$ in Ex. 3, page 8-12.)

By the effect of multiplication of t

$$L[t\, f(t)] = -\Phi'(s) = -\frac{d}{ds}\left[2(s^3+4s^2)^{-1/2}\right]$$

$$= -2\cdot\left(-\frac{1}{2}\right)\left(s^3+4s^2\right)^{-3/2}\left(3s^2+8s\right)$$

$$\therefore \quad L\left[t\, erf\, 2\sqrt{t}\right] = \frac{s(3s+8)}{s^3(s+4)^{3/2}} = \frac{3s+8}{s^2(s+4)^{3/2}}$$

Ex. 2 : Find the Laplace transforms of

(i) $t\sin at$ (ii) $(1+t\,e^{-t})^3$ **(M.U. 1997)** (iii) $t^2\sin at$

(iv) $t\,e^{-4t}\cdot\sin 3t$ (v) $t^2\, e^{-t}\sin 4t$.

Sol. : (i) $L(t\sin at) = (-1)\cdot\dfrac{d}{ds}[L(\sin at)]$

$$= -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = +\frac{2as}{(s^2+a^2)^2}$$

(ii) $L\left[(1+te^{-t})^3\right] = L\left[1 + 3t\,e^{-t} + 3t^2 e^{-2t} + t^3 e^{-3t}\right]$

$$= L(1) + 3L(t\,e^{-t}) + 3L(t^2 e^{-2t}) + L(t^3 e^{-3t})$$

$$= L(1) + 3(-1)\frac{d}{ds}\left[L(e^{-t})\right] + 3(-1)^2\frac{d^2}{ds^2}\left[L(e^{-2t})\right] + (-1)^3\frac{d^3}{ds^3}\left[L(e^{-3t})\right]$$

$$= \frac{1}{s} - 3\frac{d}{ds}\left(\frac{1}{s+1}\right) + 3\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right) - \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{s} - \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^3}$$

(iii) $L(t^2\sin at) = (-1)^2\dfrac{d^2}{ds^2}L(\sin at) = \dfrac{d^2}{ds^2}\left(\dfrac{a}{s^2+a^2}\right)$

$$= \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right] = -\frac{(s^2+a^2)^2\cdot 2a - 2as\cdot 2(s^2+a^2)\cdot 2s}{(s^2+a^2)^4}$$

$$= \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

(iv) $L\left[t\cdot e^{-4t}\sin 3t\right] = (-1)\dfrac{d}{ds}L\left[e^{-4t}\sin 3t\right]$

$$= -\frac{d}{ds}\left[\frac{3}{(s+4)^2+9}\right].$$ By shifting theorem,

$$= -\frac{d}{ds}\left[\frac{3}{s^2+8s+25}\right] = \frac{6(s+4)}{(s^2+8s+25)^2}$$

(v) $L\left[t^2\cdot e^{-t}\sin 4t\right] = (-1)^2\cdot\dfrac{d^2}{ds^2}L\left[e^{-t}\sin 4t\right]$

$$= \frac{d^2}{ds^2}\left[\frac{4}{(s+1)^2+16}\right].$$ By shifting theorem,

$$= \frac{d^2}{ds^2}\left[\frac{4}{s^2+2s+17}\right] = \frac{8(3s^2+6s-13)}{(s^2+2s+17)^3}.$$

Ex. 3 : Find the Laplace transforms of the following.

(i) $t\sin^3 t$ **(M.U. 1994)** **(ii)** $t\sin 2t\cos ht$ **(M.U. 1995)**

(iii) $t\cos^2 t$ **(M.U. 1993)** **(iv)** $t^5\cos ht$ **(M.U. 11996)**

(v) $t\,e^{3t}\sin 4t$ **(M.U. 1997)** **(vi)** $t^n e^{at}$.

Sol. : (i) $\sin 3t = 3\sin t - 4\sin^3 t$

$$L\left[\sin^3 t\right] = \frac{1}{4}\left[L(3\sin t) - L(\sin 3t)\right] = \frac{1}{4}\cdot\left[\frac{3}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$\therefore\ L\left[t\sin^3 t\right] = -\frac{3}{4}\frac{d}{ds}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

$$= -\frac{3}{4}\left[-\frac{2s}{(s^2+1)^2} + \frac{2s}{(s^2+9)^2}\right] = \frac{3s}{2}\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{s^4+18s^2+81-s^4-2s^2-1}{(s^2+1)^2(s^2+9)^2}\right] = \frac{3s}{2}\cdot\frac{16(s+5)}{(s^2+1)^2(s^2+9)^2}$$

$$= 24\cdot\frac{s(s+5)}{(s^2+1)^2(s^2+9)^2}$$

(ii) Since $\cos ht = \dfrac{e^t+e^{-t}}{2}$, $\quad \sin 2t\cos ht = \dfrac{e^t+e^{-t}}{2}\cdot\sin 2t$

Now $t\sin 2t\cos h\,t = (-1)\dfrac{d}{ds}L(\sin 2t\cos ht)$

$$= (-1)\frac{d}{ds}\left[\frac{1}{2}\left(e^t\sin 2t + e^{-t}\sin 2t\right)\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{2}{(s-1)^2+2^2} + \frac{2}{(s+1)^2+2^2}\right]$$

$$= \frac{2s-2}{(s^2-2s+5)^2} + \frac{2s+2}{(s^2+2s+5)^2}$$

$$= 2\left[\frac{s-1}{(s^2-2s+5)^2} + \frac{s+1}{(s^2+2s+5)^2}\right]$$

(iii) $L(t\cos^2 t) = L\,t\left(\dfrac{1+\cos 2t}{2}\right)$

$$= \frac{1}{2}L(t) + \frac{1}{2}L(t\cos 2t)$$

$$= \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{2}\cdot\frac{d}{ds}L(\cos 2t)$$

$$= \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{2}\frac{d}{ds}\cdot\frac{s}{s^2+2^2}$$

$$= \frac{1}{2s^2} - \frac{1}{2}\left[\frac{s^2+2^2-s\cdot 2s}{(s^2+2^2)^2}\right]$$

$$= \frac{1}{2s^2} + \frac{1}{2}\cdot\frac{s^2-2^2}{(s^2+2^2)^2}$$

(iv) $L(t^5\cos ht) = L\,t^5\left(\dfrac{e^t+e^{-t}}{2}\right) = \dfrac{1}{2}L\,t^5(e^t) + \dfrac{1}{2}L\,t^5(e^{-t})$

$$= \frac{1}{2}\cdot(-1)^5\cdot\frac{d^5}{ds^5}\left(\frac{1}{s-1}\right) + \frac{1}{2}\cdot(-1)^5\cdot\frac{d^5}{ds^5}\left(\frac{1}{s+1}\right)$$

But $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$, $\dfrac{d^2}{dx^2}\left(\dfrac{1}{x}\right) = \dfrac{2}{x^3}$

$$\frac{d^3}{dx^3}\left(\frac{1}{x}\right) = -\frac{2\cdot 3}{x^4},\ \frac{d^4}{dx^4}\left(\frac{1}{x}\right) = \frac{2\cdot 3\cdot 4}{x^5} = \frac{4!}{x^5},\ \frac{d^5}{dx^5}\left(\frac{1}{x}\right) = -\frac{5!}{x^6}$$

$$\therefore\ L(t^5\cos ht) = -\frac{1}{2}\cdot\frac{(-5!)}{(s-1)^6} - \frac{1}{2}\cdot\frac{(-5!)}{(s+1)^6}$$

$$= 60\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

(v) $L(\sin 4t) = \dfrac{4}{s^2+4^2}$ $\quad\therefore$ By first shifting theorem

$$L\left[e^{3t}\sin 4t\right] = \frac{4}{(s-3)^2+4^2} = \frac{4}{s^2-6s+25}$$

$$\therefore\ L\left[t\,e^{3t}\sin 4t\right] = -\frac{d}{ds}\left(\frac{4}{s^2-6s+25}\right)$$

$$= \frac{4(2s-6)}{(s^2-6s+25)^2} = \frac{8(s-3)}{(s^2-6s+25)^2}$$

(vi) $L\,e^{at} = \dfrac{1}{s-a}$

$$L\left[t^n e^{at}\right] = (-1)^n\cdot\frac{d^n}{ds^n}\cdot\frac{1}{(s-a)} = (-1)^n\cdot\frac{n!}{(s-a)^{n+1}}$$

Ex. 4 : Find the Laplace transforms of the following.

(i) $t\,e^{3t}\sin t$ **(M.U. 1998, 2002)**

(ii) $t\sqrt{1+\sin t}$ **(M.U. 1993, 99, 2000, 02)**

(iii) $t^3 \cos t$ **(iv)** $t\, e^{3t} erf\sqrt{t}$. **(M.U. 2000)** **(v)** $t\left(\dfrac{\sin t}{e^t}\right)^2$ **(M.U. 2004)**

Sol. : (i) We have $L \sin t = \dfrac{1}{s^2+1}$

$$\therefore \quad L(e^{3t}\sin t) = \frac{1}{(s-3)^2+1} = \frac{1}{s^2-6s+10}$$

By first shifting theorem,

$$\therefore \quad L(t\,e^{3t}\sin t) = -\frac{d}{ds}\left(\frac{1}{s^2-6s+10}\right)$$

$$= \frac{2s-6}{(s^2-6s+10)^2}.$$

(ii) we have

$$\sqrt{1+\sin t} = \sqrt{\left[\sin^2(t/2)+\cos^2(t/2)+2\sin(t/2)\cos(t/2)\right]}$$

$$= \sqrt{[\sin(t/2)+\cos(t/2)]^2} = \sin(t/2)+\cos(t/2)$$

$$\therefore \quad L\sqrt{1+\sin t} = L[\sin(t/2)+\cos(t/2)]$$

$$= \frac{1/2}{s^2+(1/2)^2} + \frac{s}{s^2+(1/2)^2} = \frac{1}{2}\cdot\frac{4}{(4s^2+1)} + \frac{4s}{(4s^2+1)}$$

$$= \frac{4s+2}{(4s^2+1)} = \frac{2(2s+1)}{(4s^2+1)}$$

$$\therefore \quad L\left[t\sqrt{1+\sin t}\right] = -\frac{d}{ds}\left[\frac{2(2s+1)}{(4s^2+1)}\right]$$

$$= -2\left[\frac{(4s^2+1)2-(2s+1)8s}{(4s^2+1)^2}\right]$$

$$= -2\frac{[-8s^2-8s+2]}{(4s^2+1)^2} = 4\frac{(4s^2+4s-1)}{(4s^2+1)^2}$$

(iii) $L\cos t = \dfrac{s}{s^2+1}$

$$\therefore \quad \frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{(s^2+1)\cdot 1 - s\cdot 2s}{(s^2+1)^2} = \frac{1-s^2}{(s^2+1)^2}$$

$$\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{(s^2+1)^2(-2s)-(1-s^2)\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}$$

$$= \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3} = \frac{2s^3-6s}{(s^2+1)^3}$$

$$\frac{d^3}{ds^3}\left(\frac{d}{s^2+1}\right) = \frac{(s^2+1)^3(6s^2-6)-(2s^3-6s)\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6}$$

$$= \frac{6\left[(s^2+1)(s^2-1)-(2s^3-6s)\,s\right]}{(s^2+1)^4} = -\frac{6(s^4-6s^2+1)}{(s^2+1)^4}$$

$$\therefore \quad L\left[t^3\cos t\right] = \frac{6(s^4-6s^2+1)}{(s^2+1)^4}.$$

(iv) We have $L(erf\sqrt{t}) = \dfrac{1}{s\sqrt{s+1}}$

$$\therefore \quad L(e^{3t} erf\sqrt{t}) = \frac{1}{(s-3)\sqrt{s-2}}$$

$$L(t\,e^{2t} erf\sqrt{t}) = -\frac{d}{ds}\left[\frac{1}{(s-3)\sqrt{s-2}}\right]$$

$$= -\left[-\frac{1}{(s-3)^2(s-2)}\frac{d}{ds}\left\{(s-3)\sqrt{s-2}\right\}\right]$$

$$= \frac{1}{(s-3)^2(s-2)}\left[(s-3)\frac{1}{2\sqrt{s-2}}+\sqrt{s-2}\right]$$

$$= \frac{1}{(s-3)^2(s-2)}\cdot\frac{(s-3+2s-4)}{2\sqrt{s-2}} = \frac{3s-7}{2(s-3)^2(s-2)^{3/2}}$$

(v) We have $t\left(\dfrac{\sin t}{e^t}\right)^2 = t\cdot e^{-2t}\sin^2 t$

$$= t\cdot e^{-2t}\left[\frac{1-\cos 2t}{2}\right] = \frac{1}{2}\cdot t\cdot e^{-2t}[1-\cos 2t]$$

Now $L(1-\cos 2t) = L(1) - L(\cos 2t)$

$$= \frac{1}{s} - \frac{s}{s^2+2^2}$$

By first shifting theorem,

$$L\{e^{-2t}(1-\cos 2t)\} = \frac{1}{s+2} - \frac{s+2}{(s+2)^2+2^2}$$

$$= \frac{1}{s+2} - \frac{s+2}{s^2+4s+8}$$

$$\therefore \quad L\left\{\frac{1}{2}\cdot t\cdot e^{-2t}(1-\cos 2t)\right\} = \frac{1}{2}\frac{d}{ds}\left\{\frac{1}{s+2} - \frac{s+2}{s^2+4s+8}\right\}$$

$$= \frac{1}{2}\left[-\frac{1}{(s+2)^2} - \frac{(s^2+4s+8)\cdot 1-(s+2)(2s+4)}{(s^2+4s+8)^2}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{(s+2)^2} + \frac{s^2+4s}{(s^2+4s+8)}\right]$$

EXERCISE

(A) 1. If $L[f(t)] = \frac{s+3}{s^2+s+1}$, find $L[t f(2t)]$. **(M.U. 2004)**

[**Ans. :** $\frac{s^2+12s+8}{(s^2+2s+4)^2}$]

2. If $L\left[erf\sqrt{t}\right] = \frac{1}{s\sqrt{s+1}}$, find $L\left[t\ erf'3\sqrt{t}\right]$. **(M.U. 2002)**

[**Ans. :** $\frac{9}{2}\cdot\frac{s+6}{s^2(s+9)^{3/2}}$]

(B) Find the Laplace transforms of the following.

1. $t^2\sin 3t$, 2. $t\cos^3 t$, 3. $t e^{-2t}\sin t$,
4. $t e^{3t}\sin 2t$, 5. $t e^{3t}\sin 3t$, **(M.U. 2004)** 6. $t e^{-3t}\cos 2t$
7. $t e^{3t}\sin^2 t$, 8. $t(2\sin 3t + e^{2t})$, 9. $t(2\sin 3t - 3\cos 3t)$,
10. $t e^{2t}(\cos t - \sin t)$, 11. $\frac{1}{4}e^{-2t}(2t\cos 2t + \sin 2t)$,
12. $t e^{t}\sin 2t\cos t$, **(M.U. 2003)** 13. $t\sqrt{1-\sin t}$, **(M.U. 2000)**
14. $t e^{-2t}\sin h\, 4t$. **(M.U. 2000)** 15. $t e^{3t}\sin h\, 2t$ **(M.U. 2003)**
16. $t\ erf\ 2\sqrt{t}$ 17. $t\sqrt{1+\sin 2t}$ **(M.U. 2003)**
18. $t\cos(wt-\alpha)$ **(M.U. 2003)** 19. $(t+\sin 2t)^2$ **(M.U. 2004)**
20. $(t\sin h\, 2t)^2$ **(M.U. 2003)** 21. $t e^{3t}\sin h\, 2t$ **(M.U. 2003)**

[**Ans. :** (1) $18\frac{(s^2-3)}{(s^2+9)^3}$, (2) $\frac{1}{4}\left[\frac{-s^2+9}{(s^2+9)^2}+\frac{s^2+3}{(s^2+1)^2}\right]$,

(3) $\frac{2(s+2)}{(s^2+4s+5)^2}$, (4) $4\cdot\frac{s-3}{(s^2-6s+13)^2}$, (5) $6\cdot\frac{(s-3)}{(s^2-6s+18)^2}$,

(6) $\frac{s^2+6s+5}{(s^2+6s+13)^2}$, (7) $\frac{1}{2}\left[\frac{1}{(s-3)^2}-\frac{(s^2-6s+5)}{(s^2-6s+13)^2}\right]$,

(8) $\frac{12s}{(s^2+9)^2}+\frac{1}{(s-2)^2}$, (9) $\frac{3(9+4s-s^2)}{(s^2+9)^2}$, (10) $\frac{s^2-6s+7}{(s^2-4s+5)^2}$,

(11) $\frac{(s+2)^2}{(s^2+4s+8)^2}$, (12) $\frac{3(s-1)}{(s^2-2s+10)^2}+\frac{s-1}{(s^2-2s+2)}$,

(13) $4\cdot\frac{(4s^2-4s-1)}{(4s^2+1)^2}$, (14) $\frac{8(s+2)}{(s^2+4s-12)^2}$, (15) $\frac{4(s-3)}{(s^2-6s+5)^2}$

(16) $\frac{3s+8}{s^2(s+4)^{3/2}}$, (17) $\frac{s^2+2s-1}{(s^2+1)^2}$, (18) $\frac{(s^2-w^2)\cos\alpha+2ws\sin\alpha}{(s^2+w^2)^2}$,

(19) $\frac{2}{s^3}+\frac{s}{(s^2+1)^2}+\frac{1}{2s}-\frac{s}{2(s^2+4)}$, (20) $\frac{1}{2}\left[\frac{1}{(s-4)^3}+\frac{1}{(s+4)^3}\right]$,

(21) $\frac{1}{2}\left[\frac{1}{(s-5)^2}-\frac{1}{(s-1)^2}\right]$

(a) Particular value of the function $L[t f(t)]$

Ex. 1 : Find $\int_0^\infty e^{-3t} t\sin t\cdot dt$. **(M.U. 2002)**

Sol. : Consider $\int_0^\infty e^{-st} t\sin t\cdot dt = L[t\sin t]$

$$= (-1)\frac{d}{ds}[L(\sin t)] = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

Putting $s = 3$, $\int_0^\infty e^{-3t}\cdot t\sin t\cdot dt = \frac{6}{(9+1)^2} = \frac{3}{50}$.

Ex. 2 : Find $\int_0^\infty e^{-3t}\, t\cos t\cdot dt$.

Sol. : Consider $\int_0^\infty e^{-st}\, t\cos t\cdot dt = L(t\cos t) = -\frac{d}{ds}L(\cos t)$

$$= -\frac{d}{ds}\left[\frac{s}{s^2+1}\right] = -\left[\frac{(s^2+1)-s\cdot 2s}{(s^2+1)^2}\right] = \frac{s^2-1}{(s^2+1)^3}$$

Putting $s = 3$, $\int_0^\infty e^{-3t}\, t\cos t\cdot dt = \frac{8}{100} = \frac{2}{25}$.

Ex. 3 : Find $\int_0^\infty e^{-t}\, t^3\sin t\, dt$. **(M.U. 1996)**

Sol. : By definition $\int_0^\infty e^{-st}\, t^3\sin t\, dt = L(t^3\sin t)$

$$= (-1)^3\frac{d^3}{ds^3}L(\sin t) = (-1)\frac{d^3}{ds^3}\left[\frac{1}{s^2+1}\right]$$

Now $\dfrac{d}{ds}\left[\dfrac{1}{s^2+1}\right]=-\dfrac{2s}{(s^2+1)^2}$

$$\frac{d}{ds}\left[-\frac{2s}{(s^2+1)^2}\right]=-2\left[\frac{(s^2+1)^2\cdot 1-s\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}\right]$$

$$=-2\left[\frac{(s^2+1)-4s^2}{(s^2+1)^3}\right]=-2\frac{(-3s^2+1)}{(s^2+1)^3}=2\frac{(3s^2-1)}{(s^2+1)^3}$$

$$\frac{d}{ds}\left[2\frac{(3s^2-1)}{(s^2+1)^3}\right]=2\left[\frac{(s^2+1)^3\cdot 6s-(3s^2-1)\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6}\right]$$

$$=2\left[\frac{6s(s^2+1)-6s(3s^2-1)}{(s^2+1)^4}\right]=-24s\cdot\frac{(s^2-1)}{(s^2+1)^4}$$

Thus, we have

$$\int_0^\infty e^{-st}\,t^3\sin t\,dt=24s\cdot\frac{(s^2-1)}{(s^2+1)^4}$$

To find the value of the given integral we put $s=1$.

$$\therefore\quad \int_0^\infty e^{-t}\,t^3\sin t\,dt=0.$$

Ex. 4 : If $L[J_o(t)]=\dfrac{1}{\sqrt{1+s^2}}$, prove that

(i) $\int_0^\infty J_o(t)\,dt=1.$ **(ii)** $L[t\,J_o(at)]=\dfrac{s}{(s^2+a^2)^{3/2}}.$

(iii) $L\left[e^{-bt}J_o(at)\right]=\dfrac{1}{\sqrt{(s+b)^2+a^2}}.$ **(M.U. 1998)**

(iv) $\int_0^\infty t\,e^{-3t}J_o(4t)\,dt=\dfrac{3}{125}.$ **(M.U. 1996, 2003)**

Sol. : (i) By definition of Laplace transform since $L[J_o(t)]=\dfrac{1}{\sqrt{1+s^2}}$,

$$\int_0^\infty e^{-st}J_o(t)\,dt=\frac{1}{\sqrt{1+s^2}}.$$

Putting $s=0$, we get $\int_0^\infty J_o(t)\,dt=\dfrac{1}{\sqrt{1+0}}=\dfrac{1}{1}=1.$

(ii) Since $J_o(t)=\dfrac{1}{\sqrt{1+s^2}}$, by change of scale property (§ 5 page 3-16).

$$J_o(at)=\frac{1}{a}\cdot\frac{1}{\sqrt{1+(s/a)^2}}=\frac{1}{\sqrt{s^2+a^2}}$$

$$\therefore\quad L[t\,J_o(at)]=(-1)\frac{d}{ds}\left(\frac{1}{\sqrt{s^2+a^2}}\right)\qquad\text{(by § 8 page 3-25)}$$

$$=(-1)\left(-\frac{1}{2}\right)(s^2+a^2)^{-3/2}\cdot 2s=\frac{s}{(s^2+a^2)^{3/2}}$$

(iii) Since $J_o(at)=\dfrac{1}{\sqrt{s^2+a^2}}$, by first shifting theorem (§ 6 page 3-18).

$$L\left[e^{-bt}J_o(at)\right]=\frac{1}{\sqrt{(s+b)^2+a^2}}.$$

(iv) Now putting $a=4$ in (2) above $L[t\,J_o(4t)]=\dfrac{s}{(s^2+16)^{3/2}}.$

By definition of Laplace transform this equation means

$$\int_0^\infty e^{-st}\,t\,J_o(4t)\,dt=\frac{s}{(s^2+16)^{3/2}}$$

Putting $s=3$, we get $\int_0^\infty e^{-3t}\,t\,J_o(4t)\,dt=\dfrac{3}{(9+16)^{3/2}}=\dfrac{3}{125}.$

Ex. 5 : Evaluate $\int_0^\infty \dfrac{t^2\sin 3t}{e^{2t}}\,dt.$ **(M.U. 2002)**

Sol. : Consider $\int_0^\infty e^{-st}\,t^2\sin 3t\,dt=L(t^2\sin 3t)$

$$=(-1)^2\frac{d^2}{ds^2}[L(\sin 3t)]$$

$$=\frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)=\frac{d}{ds}\left(-\frac{3\cdot 2s}{(s^2+9)^2}\right)$$

$$=-6\left[\frac{(s^2+9)^2\cdot 1-s\cdot 2(s^2+9)\cdot 2s}{(s^2+9)^4}\right]$$

$$=-6\left[\frac{s^2+9-4s^2}{(s^2+9)^3}\right]=\frac{-6\cdot(-3s^2+9)}{(s^2+9)^3}$$

$$=\frac{18(s^2-3)}{(s^2+9)^3}$$

Putting $s=2$ in the above equality,

$$\int_0^\infty e^{-2t}\,t^2\sin 3t\,dt=\frac{18(4-3)}{(4+9)^3}=\frac{18}{2197}.$$

EXERCISE

(A) Evaluate the following using Laplace transform.

1. $\int_0^\infty e^{-2t}\, t^3 \sin t\, dt.$ 2. $\int_0^\infty e^{-2t}\, t \cos t\, dt.$

3. $\int_0^\infty e^{-2t}\, t \sin^2 t\, dt.$ 4. $\int_0^\infty e^{-3t}\, t^2 \sin h\, 2t\, dt.$

[**Ans. :** (1) $-\frac{576}{25}$, (2) $\frac{3}{25}$, (3) $\frac{1}{8}$, (4) $\frac{124}{125}$.]

(B) If $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$, then evaluate the following

1. $\int_0^\infty e^{-t} J_0(t)\, dt.$ 2. $\int_0^\infty e^{-4t} t\, J_0(3t)\, dt.$ [**Ans.** (1) $\frac{1}{\sqrt{2}}$, (2) $\frac{4}{125}$.]

(C) If $L(erf\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$, evaluate $\int_0^\infty e^{-2t} erf(2\sqrt{t})\, dt.$ **(M.U. 2003)**

[**Ans. :** $1/\sqrt{6}$]

9. Effect Of Division By t

If $L[f(t)] = \Phi(s)$, then $\boxed{L\left[\frac{1}{t} f(t)\right] = \int_s^\infty \Phi(s)\, ds}$ .... (15)

(M.U. 1994)

Proof : We have $\Phi(s) = \int_0^\infty e^{-st} f(t)\, dt$

Integrating both sides w.r.t. s between the limits s, ∞ and changing the order of integration on r.h.s.

$$\int_s^\infty \Phi(s)\, ds = \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\, ds\right] dt$$

$$= \int_0^\infty \left[\frac{e^{-st}}{-t} f(t)\right]_s^\infty dt = \int_s^\infty e^{-st} \frac{f(t)}{t}\, dt = L\left[\frac{1}{t} f(t)\right]$$

Ex. 1 : Find the Laplace transform of the following.

(i) $\frac{1}{t}(1-\cos t)$ **(M.U. 1995, 2004)** **(ii)** $\frac{1}{t} e^{-t} \sin t$ **(M.U. 2003)**

(iii) $\frac{1}{t}(e^{-at} - e^{-bt})$ **(M.U. 1997)**

Sol. : (i) $L[1-\cos t] = L(1) - L(\cos t) = \frac{1}{s} - \frac{s}{s^2+1}$

$$\therefore\ L\left[\frac{1}{t}(1-\cos t)\right] = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds = \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= -\frac{1}{2}\left[\log(s^2+1) - \log s^2\right]_s^\infty$$

$$= -\frac{1}{2}\left[\log\frac{s^2+1}{s^2}\right]_s^\infty = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

(ii) $L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1}$

$$\therefore\ L\left[\frac{1}{t}(e^{-t}\sin t)\right] = \int_s^\infty \frac{1}{(s+1)^2+1}\, ds = \left[\tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

(iii) $L\left(e^{-at} - e^{-bt}\right) = \frac{1}{s+a} - \frac{1}{s+b}$

$$\therefore\ L\left[\frac{1}{t}\left(e^{-at} - e^{-bt}\right)\right] = \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right] ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left\{\frac{s+a}{s+b}\right\}\right]_s^\infty = -\log\left(\frac{s+a}{s+b}\right) = \log\left(\frac{s+b}{s+a}\right)$$

Ex. 2 : Find the Laplace transform of the following.

(i) $\frac{\sin^2 2t}{t}$ **(M.U. 1993)** **(ii)** $\frac{1-\cos t}{t^2}$

(iii) $\frac{e^{-2t}\sin 2t \cos h\, t}{t}$ **(M.U. 1996)**

Sol. : (i) $\sin^2 2t = \frac{1-\cos 4t}{2}$.

$$\therefore\ L\sin^2 2t = L\frac{1}{2}(1-\cos 4t) = \frac{1}{2}[L(1) - L(\cos 4t)]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right]$$

$$\therefore\ L\left[\frac{\sin^2 2t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4^2}\right] ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2 + 4^2)\right]_s^\infty$$

$$= -\frac{1}{4}\left[\log(s^2 + 4^2) - \log s^2\right]_s^\infty$$

$$= -\frac{1}{4}\left[\log\left(\frac{s^2 + 4^2}{s^2}\right)\right]_s^\infty = \frac{1}{4}\log\left(\frac{s^2 + 4^2}{s^2}\right).$$

(ii) $L[1 - \cos t] = L(1) - L(\cos t) = \frac{1}{s} - \frac{s}{s^2 + 1}$

$$\therefore \quad L\left(\frac{1 - \cos t}{t}\right) = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2 + 1}\right] ds = \left[\log s - \frac{1}{2}\log(s^2 + 1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2 + 1}\right)\right]_s^\infty = -\frac{1}{2}\log\frac{s^2}{s^2 + 1} = \frac{1}{2}\log\left(\frac{s^2 + 1}{s^2}\right)$$

$$\therefore \quad L\left(\frac{1 - \cos t}{t^2}\right) = \int_s^\infty \frac{1}{2}\log\left(\frac{s^2 + 1}{s^2}\right) ds$$

Integrating by parts

$$L\left(\frac{1 - \cos t}{t^2}\right) = \frac{1}{2}\left[\log\left(\frac{s^2 + 1}{s^2}\right) \cdot s - \int s \cdot \left(\frac{s^2}{s^2 + 1}\right)\left(\frac{s^2 \cdot 2s - (s^2 + 1)2s}{s^4}\right) ds\right]_s^\infty$$

$$= \frac{1}{2}\left[s\log\left(\frac{s^2 + 1}{s^2}\right) + 2\int \frac{ds}{s^2 + 1}\right]_s^\infty = \frac{1}{2}\left[s\log\left(\frac{s^2 + 1}{s^2}\right) + 2\tan^{-1} s\right]_s^\infty$$

$$= \frac{1}{2}\left[0 + 2 \cdot \frac{\pi}{2} - s\log\left(\frac{s^2 + 1}{s^2}\right) - 2\tan^{-1} s\right]$$

$$= \frac{\pi}{2} - \frac{s}{2}\log\left(\frac{s^2 + 1}{s^2}\right) - \tan^{-1} s.$$

(iii) We have, $e^{-2t}\sin 2t \cos ht = e^{-2t}\sin 2t \frac{(e^t + e^{-t})}{2}$

$$= \frac{1}{2}\left(e^{-t}\sin 2t + e^{-3t}\sin 2t\right)$$

$$\therefore \quad L\left[e^{-2t}\sin 2t \cos h\, t\right] = \frac{1}{2}L\left(e^{-t}\sin 2t + e^{-3t}\sin 2t\right)$$

But $L \sin 2t = \frac{2}{s^2 + 2^2}$

$$\therefore \quad L\left(e^{-t}\sin 2t\right) = \frac{2}{(s + 1)^2 + 2^2} = \frac{2}{s^2 + 2s + 5}$$

$$L\left(e^{-3t}\sin 2t\right) = \frac{2}{(s + 3)^2 + 2^2} = \frac{2}{s^2 + 6s + 13}$$

$$\therefore \quad L\left[e^{-2t}\sin 2t \cos h\, t\right] = \frac{1}{s^2 + 2s + 5} + \frac{1}{s^2 + 6s + 13}$$

$$\therefore \quad L\left[\frac{e^{-2t}\sin 2t \cos ht}{t}\right] = \int_s^\infty \left(\frac{1}{s^2 + 2s + 5} + \frac{1}{s^2 + 6s + 13}\right) ds$$

$$= \int_s^\infty \left[\frac{1}{(s + 1)^2 + 2^2} + \frac{1}{(s + 3)^2 + 2^2}\right] ds$$

$$= \left[\frac{1}{2}\tan^{-1}\frac{s + 1}{2} + \frac{1}{2}\tan^{-1}\frac{s + 3}{2}\right]_s^\infty$$

$$= \frac{1}{2} \cdot \left(\frac{\pi}{2} - \tan^{-1}\frac{s + 1}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}\frac{s + 3}{2}\right).$$

$$= \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{s + 1}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{s + 3}{2}\right).$$

Ex. 3 : Find the Laplace transform of $\frac{\sin at}{t}$. Does Laplace transform of $\frac{\cos at}{t}$ exist ?

Sol. : Consider $f(t) = \sin at$

$$\therefore \quad L[f(t)] = \frac{a}{s^2 + a^2} = \Phi(s)$$

$$\therefore \quad L\left[\frac{\sin at}{t}\right] = \int_s^\infty \Phi(s)\, ds \qquad \text{[By (15) page 3-36]}$$

$$= \int_s^\infty \frac{a}{s^2 + a^2} ds = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} = \cot^{-1}\frac{s}{a}.$$

Now consider, $f(t) = \text{cost } at$

$$\therefore \quad L[f(t)] = \frac{s}{s^2 + a^2} = \Phi(s)$$

$$\therefore \quad L\left[\frac{\cos at}{t}\right] = \int_s^\infty \Phi(s)\, ds$$

$$= \int_s^\infty \frac{s}{s^2+a^2}\,ds = \frac{1}{2}\left[\log(s^2+a^2)\right]_s^\infty$$

Since $\log(s^2+a^2)$ is infinite when $s \to \infty$, $L\left[\frac{\cos at}{t}\right]$ does not exist.

Ex. 4 : Find $L\left[\frac{\sin^2 t}{t^2}\right]$. **(M.U. 2005)**

Sol. : $L(\sin^2 t) = L\left[\frac{1-\cos 2t}{2}\right]$

$$= \frac{1}{2}[L(1) - L\cos 2t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$\therefore \quad L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty = \frac{1}{4}\left[\log\frac{s^2}{s^2+4}\right]_s^\infty$$

$$= -\frac{1}{4}\log\left(\frac{s^2}{s^2+4}\right) = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

$$\therefore \quad L\left[\frac{\sin^2 t}{t^2}\right] = \int_s^\infty \frac{1}{4}\cdot\log\left(\frac{s^2+4}{s^2}\right) ds$$

Integrating by parts,

$$L\left[\frac{\sin^2 t}{t^2}\right] = \frac{1}{4}\left[\log\left(\frac{s^2+4}{s^2}\right)\cdot s - \int s\cdot\frac{s^2}{s^2+4}\left(\frac{s^2\cdot 2s-(s^2+4)\cdot 2s}{s^4}\right)ds\right]_s^\infty$$

$$= \frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right) + 8\int\frac{ds}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right) + 2\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[0 + 2\cdot\frac{\pi}{2} - s\log\left(\frac{s^2+4}{s^2}\right) - 2\tan^{-1}\frac{s}{2}\right]$$

$$= \frac{\pi}{4} - \frac{s}{4}\log\left(\frac{s^2+4}{s^2}\right) - \frac{1}{2}\cdot\tan^{-1}\left(\frac{s}{2}\right).$$

EXERCISE

Find the Laplace transform of the following.

1. $\frac{\sin t}{t}$ **(M.U. 1994)**
2. $\frac{1}{t}[1-\cos at]$
3. $\frac{1}{t}[e^{-t}\sin at]$
4. $\frac{1}{t}[1-e^{2t}]$
5. $\frac{1}{t}[e^{-3t}\sin 2t]$
6. $\frac{1}{t}[\sin^2 t]$
7. $\frac{1}{t}[e^{2t}\sin^3 t]$
8. $\frac{1}{t}[\cos at - \cos bt]$
9. $\frac{1}{t}[1-\cos 3t]$
10. $\frac{1}{t}[e^{-2t}\sin 3t]$
11. $\frac{1}{t}(\sin^3 t)$
12. $\frac{1}{t}[e^{-t}\sin^3 t]$
13. $\frac{e^{2t}\sin t}{t}$ **(M.U. 1999)**
14. $\frac{\cosh 2t\sin 2t}{t}$ **(M.U. 1999)**
15. $\frac{2\sin t\sin 2t}{t}$ **(M.U. 2004)**
16. $\frac{\sin t\sin 5t}{t}$ **(M.U. 2003)**
17. $\frac{\sin^2 t}{t}$ **(M.U. 2003)**
18. $\frac{(1-\cos 2t)}{t}$ **(M.U. 2003)**
19. $\frac{\sinh at}{t}$ **(M.U. 2004)**

[**Ans. : (1)** $\cot^{-1} s$, **(2)** $\frac{1}{2}\log\frac{(s^2+a^2)}{s^2}$, **(3)** $\cot^{-1}\left(\frac{s+1}{a}\right)$,

(4) $\log\left(\frac{s-2}{s}\right)$, **5)** $\cot^{-1}\left(\frac{s-3}{2}\right)$, **(6)** $\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$,

(7) $\frac{3}{4}\cot^{-1}(s-2) - \frac{1}{4}\cot^{-1}\left(\frac{s-2}{3}\right)$, **(8)** $\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$,

(9) $\frac{1}{2}\log\left(\frac{s^2+9}{s^2}\right)$, **(10)** $\cot^{-1}\left(\frac{s+2}{3}\right)$,

(11) $\frac{1}{4}\left[3\cot^{-1}s - \cot^{-1}\frac{s}{3}\right]$, **(12)** $\frac{1}{4}\left[3\cot^{-1}(s+1) - \cot^{-1}\left(\frac{s+1}{3}\right)\right]$,

(13) $\cot^{-1}(s-2)$, **(14)** $\pi + \tan^{-1}\left(\frac{s-2}{2}\right) + \tan^{-1}\left(\frac{s+2}{2}\right)$,

(15) $\frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$ **(16)** $\frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)$

(17) $\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$ **(18)** $\frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$ **(19)** $\frac{1}{2}\log\left(\frac{s-a}{s+a}\right)$.

(a) Particular Value of the function $L\left[\frac{f(t)}{t}\right]$

Ex. 1 : Evaluate $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt.$

Sol. : Consider $f(t) = e^{-at} - e^{-bt}$

$$\therefore \quad L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$\therefore \quad L\left[\frac{1}{t}f(t)\right] = -\left[\log\left(\frac{s+b}{s+a}\right)\right]_s^\infty = \log\left(\frac{s+b}{s+a}\right) \qquad \text{.......... (1)}$$

The equation (1) means

$$\int_0^\infty e^{-st}\left(\frac{e^{-at}-e^{-bt}}{t}\right) dt = \log\left(\frac{s+b}{s+a}\right)$$

Putting $s = 0$, $\quad \int_0^\infty \left(\frac{e^{-at}-e^{-bt}}{t}\right) dt = \log\frac{b}{a}$

Ex. 2 : Evaluate $\int_0^\infty \frac{\cos at - \cos bt}{t}\,dt.$ **(M.U. 2004)**

Sol. : Consider $f(t) = \cos at - \cos bt$

$$\therefore \quad L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= -\frac{1}{2}\left[\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right]_s^\infty = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right) \qquad \text{.......... (1)}$$

By definition of Laplace transform the equation (1) means

$$\int_0^\infty e^{-st}\left(\frac{\cos at - \cos bt}{t}\right) dt = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Putting $s = 0$,

$$\int_0^\infty \frac{\cos at - \cos bt}{t}\,dt = \frac{1}{2}\log\frac{b^2}{a^2} = \log\frac{b}{a}.$$

Ex. 3 : Evaluate $\int_0^\infty e^{-st} \cdot \frac{\sin^2(at/2)}{t} \cdot dt.$

Sol. : We have $\quad \sin^2\left(\frac{at}{2}\right) = \frac{1-\cos at}{2}$

$$\therefore \quad L(\sin^2 at) = \frac{1}{2}L(1) - \frac{1}{2}L(\cos at) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{s}{(s^2+a^2)}$$

$$\therefore \quad L\left(\frac{\sin^2 at}{t}\right) = \int_s^\infty \frac{1}{2}\cdot\frac{1}{s}\,ds - \int_s^\infty \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2s}{s^2+a^2}\,ds$$

$$= \left[\frac{1}{2}\log s - \frac{1}{4}\log(s^2+a^2)\right]_s^\infty = \frac{1}{4}\left[\log\frac{s^2}{s^2+a^2}\right]_s^\infty$$

$$= -\frac{1}{4}\log\left(\frac{s^2}{s^2+a^2}\right) = \frac{1}{4}\log\left(\frac{s^2+a^2}{s^2}\right)$$

This means $\quad \int_0^\infty e^{-st} \cdot \frac{\sin^2 at}{t}\,dt = \frac{1}{4}\log\left(\frac{s^2+a^2}{s^2}\right).$

Ex. 4 : Prove that $\int_0^\infty e^{-t} \cdot \frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log 5.$

(M.U. 1998, 99, 2000, 03, 04, 05)

Sol. : In the above example put $s = 1$, $a = 2$.

$$\therefore \quad \int_0^\infty e^{-t} \cdot \frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log\left(\frac{1+4}{1}\right) = \frac{1}{4}\log 5$$

Or independently, consider

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

$$\therefore \quad L\sin^2 t = \frac{1}{2}L(1) - \frac{1}{2}L(\cos 2t) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s}{s^2+4}$$

$$\therefore \quad L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty \frac{1}{2}\cdot\frac{1}{s}\,ds - \int_s^\infty \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2s}{s^2+4}\,ds$$

$$= \left[\frac{1}{2}\log s - \frac{1}{4}\log(s^2+4)\right]_s^\infty = \frac{1}{4}\left[\log\frac{s^2}{s^2+4}\right]_s^\infty$$

$$= -\frac{1}{4}\log\frac{s^2}{s^2+4} = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right).$$

This means $\quad \int_0^\infty e^{-st} \cdot \frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

Now put $s = 1$, $\quad \therefore \int_0^\infty e^{-t} \cdot \frac{\sin^2 t}{t}\, dt = \frac{1}{4}\log 5.$

Ex. 5 : Prove that $\int_0^\infty e^{-st} \cdot \frac{\sin t \sin h\, at}{t} = \frac{1}{2}\tan^{-1}\left[\frac{2a}{1+(s^2-a^2)}\right].$

Sol. : We have, $\sin t \sin h\, at = \left(\frac{e^{at}-e^{-at}}{2}\right)\sin t$

Now $\quad L \sin t = \frac{1}{s^2+1}$

$$\therefore \quad L\left(e^{at}\sin t\right) = \frac{1}{(s-a)^2+1}, \quad L\left(e^{-at}\sin t\right) = \frac{1}{(s+a)^2+1}$$

$$\therefore \quad L \sin t \sin h\, at = \frac{1}{2}\left[\frac{1}{(s-a)^2+1} - \frac{1}{(s+a)^2+1}\right]$$

$$\therefore \quad L\left(\frac{\sin t \sin h\, at}{t}\right) = \frac{1}{2}\int_s^\infty \left[\frac{1}{(s-a)^2+1} - \frac{1}{(s+a)^2+1}\right] ds$$

$$= \frac{1}{2}\left[\tan^{-1}(s-a) - \tan^{-1}(s+a)\right]_s^\infty$$

$$= \frac{1}{2}\left[\left\{\frac{\pi}{2} - \tan^{-1}(s-a)\right\} - \left\{\frac{\pi}{2} - \tan^{-1}(s+a)\right\}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+a) - \tan^{-1}(s-a)\right]$$

Now let $\tan^{-1}(s+a) = \alpha$, $\quad \tan^{-1}(s-a) = \beta$

$\therefore \quad \tan\alpha = s+a, \quad \tan\beta = s-a$

$$\therefore \quad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$= \frac{(s+a)-(s-a)}{1+(s^2-a^2)} = \frac{2a}{1+(s^2-a^2)}.$$

$$\therefore \quad \alpha - \beta = \tan^{-1}\frac{2a}{1+(s^2+a^1)}$$

$$\therefore \quad \tan^{-1}(s+a) - \tan^{-1}(s-a) = \tan^{-1}\frac{2a}{1+(s^2-a^2)}.$$

$$\therefore \quad L\left(\frac{\sin t \sin h\, at}{t}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2a}{1+(s^2-a^2)}\right)$$

This means $\quad \int_0^\infty e^{-st} \cdot \frac{\sin t \sin h\, at}{t}\, dt = \frac{1}{2}\tan^{-1}\left(\frac{2a}{1+(s^2-a^2)}\right).$

Ex. 6 : Prove that $\int_0^\infty e^{-\sqrt{2}\,t} \sin t \sin h\, t = \frac{\pi}{8}.$ **(M.U. 2002)**

Sol. : In the above example put $s = \sqrt{2}$, $a = 1$.

$$\therefore \quad \int_0^\infty e^{-\sqrt{2}\,t} \sin t \sin h\, t\, dt = \frac{1}{2}\tan^{-1}\left(\frac{2}{1+(2-1)}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2}{2}\right) = \frac{1}{2}\tan^{-1} 1 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$

Or independently consider

$$\sin t \sin h\, t = \left(\frac{e^t - e^{-t}}{2}\right)\sin t.$$

Now, $\quad L \sin t = \frac{1}{t^2+1}$

$$\therefore \quad L\left(e^t \sin t\right) = \frac{1}{(s-1)^2+1}, \quad L\left(e^{-t}\sin t\right) = \frac{1}{(s+1)^2+1}.$$

$$\therefore \quad L \sin t \sin h\, t = \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore \quad L\left(\frac{\sin t \sin h\, t}{t}\right) = \frac{1}{2}\int_s^\infty \left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right] ds$$

$$= \frac{1}{2}\left[\tan^{-1}(s-1) - \tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\left\{\frac{\pi}{2} - \tan^{-1}(s-1)\right\} - \left\{\frac{\pi}{2} - \tan^{-1}(s+1)\right\}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right].$$

Now let $\tan^{-1}(s+1) = \alpha$, $\quad \tan^{-1}(s-1) = \beta$.

$\therefore \quad \tan\alpha = s+1, \quad \tan\beta = s-1$

$$\therefore \quad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\alpha}$$

$$= \frac{(s+1)-(s-1)}{1+(s^2-1)} = \frac{2}{1+(s^2-1)} = \frac{2}{s^2}$$

$$\therefore \quad \alpha - \beta = \tan^{-1}\left(\frac{2}{s^2}\right)$$

$$\therefore \quad \tan^{-1}(s+1) - \tan^{-1}(s-1) = \tan^{-1}\frac{2}{s^2}$$

$$\therefore \quad L\left(\frac{\sin t \sinh t}{t}\right)=\frac{1}{2}\tan^{-1}\frac{2}{s^2}$$

This means $\displaystyle\int_0^\infty e^{-st}\cdot\frac{\sin t\sinh t}{t}=\frac{1}{2}\tan^{-1}\frac{2}{s^2}$

Now put $s=\sqrt{2}$

$$\therefore \quad \int_0^\infty e^{-\sqrt{2}\,t}\left(\frac{\sin t\sinh t}{t}\right)=\frac{1}{2}\tan^{-1}\frac{2}{2}$$

$$=\frac{1}{2}\tan^{-1}1=\frac{1}{2}\cdot\frac{\pi}{4}=\frac{\pi}{8}.$$

Ex. 7 : Prove that $\displaystyle\int_0^\infty e^{-st}\left(\frac{\sin at+\sin bt}{t}\right)dt=\pi-\tan^{-1}\left[\frac{s(a+b)}{ab-s^2}\right]$.

Sol. : We have $L(\sin at)=\dfrac{a}{s^2+a^2}$, $L(\sin bt)=\dfrac{b}{s^2+b^2}$

$$\therefore \quad L\left(\frac{\sin at+\sin bt}{t}\right)=\int_s^\infty\frac{a}{s^2+a^2}\,ds+\int_s^\infty\frac{b}{s^2+b^2}\,ds$$

$$=\left[\tan^{-1}\frac{s}{a}\right]_s^\infty+\left[\tan^{-1}\frac{s}{b}\right]_s^\infty$$

$$=\frac{\pi}{2}-\tan^{-1}\frac{s}{a}+\frac{\pi}{2}-\tan^{-1}\frac{s}{b}$$

$$=\pi-\left(\tan^{-1}\frac{s}{a}+\tan^{-1}\frac{s}{b}\right)$$

Now let $\tan^{-1}\dfrac{s}{a}=\alpha$, $\tan^{-1}\dfrac{s}{b}=\beta$ $\therefore$ $\tan\alpha=\dfrac{s}{a}$, $\tan\beta=\dfrac{s}{b}$

$$\therefore \quad \tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$=\frac{(s/a)+(s/b)}{1-(s/a)(s/b)}=\frac{s(a+b)}{ab-s^2}$$

$$\alpha+\beta=\tan^{-1}\left[\frac{s(a+b)}{ab-s^2}\right]$$

$$\therefore \quad L\left(\frac{\sin at+\sin bt}{t}\right)=\pi-\tan^{-1}\left[\frac{s(a+b)}{ab-s^2}\right]$$

This means $\displaystyle\int_0^\infty e^{-st}\cdot\left(\frac{\sin at+\sin bt}{t}\right)dt=\pi-\tan^{-1}\left[\frac{s(a+b)}{ab-s^2}\right]$.

Ex. 8 : Find the Laplace transform of $\dfrac{e^{-at}-\cos at}{t}$.

Hence, evaluate $\displaystyle\int_0^\infty\frac{e^{-t}-\cos t}{t\,e^{4t}}\,dt.$ **(M.U. 2004)**

Sol. : We have $L(e^{-at})=\dfrac{1}{s+a}$ and $L(\cos at)=\dfrac{s}{s^2+a^2}$

$$\therefore \quad L[e^{-at}-\cos at]=\frac{1}{s+a}-\frac{s}{s^2+a^2}$$

$$\therefore \quad L\left[\frac{e^{-at}-\cos at}{t}\right]=\int_s^\infty\left(\frac{1}{s+a}-\frac{s}{s^2+a^2}\right)ds$$

$$=\left[\log(s+a)-\frac{1}{2}\log(s^2+a^2)\right]_s^\infty$$

$$=\left[\log\left(\frac{s+a}{\sqrt{s^2+a^2}}\right)\right]_s^\infty=\left[\log\left(\frac{1+(a/s)}{\sqrt{1+(a^2/s^2)}}\right)\right]_s^\infty$$

$$=\log 1-\log\left(\frac{1+(a/s)}{\sqrt{1+(a^2/s^2)}}\right)=\log\left(\frac{\sqrt{s^2+a^2}}{s+a}\right)$$

$$\therefore \quad \int_0^\infty e^{-st}\left(\frac{e^{-at}-\cos at}{t}\right)dt=\log\left(\frac{\sqrt{s^2+a^2}}{s+a}\right) \qquad \text{.......... (1)}$$

Now, $\displaystyle\int_0^\infty\frac{e^{-t}-\cos t}{t\,e^{4t}}\,dt=\int_0^\infty e^{-4t}\left(\frac{e^{t}-\cos t}{t}\right)dt$

Putting $s=4$ and $a=1$ in (1), we get

$$\int_0^\infty\frac{e^{-t}-\cos t}{t\,e^{4t}}\,dt=\log\left(\frac{\sqrt{16+1}}{4}\right)=\log\frac{\sqrt{17}}{5}$$

Ex. 9 : Prove that $\displaystyle\int_0^\infty\left(\frac{\sin 2t+\sin 3t}{t\,e^{t}}\right)dt=\frac{3\pi}{4}$. **(M.U. 1995)**

Sol. : In Ex. 7 above put $s=1$, $a=2$, $b=3$.

$$\therefore \quad \int_0^\infty e^{-t}\cdot\left(\frac{\sin 2t+\sin 3t}{t}\right)dt$$

$$=\pi-\tan^{-1}\left[\frac{(1)(5)}{6-1}\right]=\pi-\tan^{-1}\left(\frac{5}{5}\right)$$

$$=\pi-\tan^{-1}(1)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}.$$

Or independently, we have,

$$L(\sin 2t) = \frac{2}{s^2+4}, \quad L(\sin 3t) = \frac{3}{s^2+9}$$

$$\therefore \quad L\left(\frac{\sin 2t + \sin 3t}{t}\right) = \int_s^\infty \left[\frac{2}{(s^2+4)} + \frac{3}{(s^2+9)}\right] ds$$

$$= \left[\tan^{-1}(s/2)\right]_s^\infty + \left[\tan^{-1}(s/3)\right]_s^\infty$$

$$= \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{2}\right) + \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right)$$

$$= \pi - \left(\tan^{-1}\frac{s}{2} + \tan^{-1}\frac{s}{3}\right)$$

Now let $\tan^{-1}\frac{s}{2} = \alpha, \quad \tan^{-1}\frac{s}{3} = \beta \quad \therefore \quad \frac{s}{2} = \tan\alpha, \quad \frac{s}{3} = \tan\beta$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{(s/2)+(s/3)}{1-(s/2)(s/3)} = \frac{5s}{6-s^2}$$

$$\therefore \quad \alpha + \beta = \tan^{-1}\left(\frac{5s}{6-s^2}\right)$$

$$\therefore \quad L\left(\frac{\sin 2t + \sin 3t}{t}\right) = \pi - \tan^{-1}\left(\frac{5s}{6-s^2}\right)$$

This means $\int_0^\infty e^{-st} \cdot \left(\frac{\sin 2t + \sin 3t}{t}\right) dt = \pi - \tan^{-1}\left(\frac{5s}{6-s^2}\right)$

Now put $s = 1$,

$$\int_0^\infty e^{-t} \cdot \left(\frac{\sin 2t + \sin 3t}{t}\right) dt = \pi - \tan^{-1}\left(\frac{5}{6-1}\right)$$

$$= \pi - \tan^{-1}\left(\frac{5}{5}\right) = \pi - \tan^{-1}(1)$$

$$= \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Ex. 10 : Show that $\int_0^\infty \frac{\sin at}{t} dt = \frac{\pi}{2}$. **(M.U. 1997)**

Sol. : We have proved in Ex. 3 page 3-39 the result.

$$L\left[\frac{\sin at}{t}\right] = \cot^{-1}\frac{s}{a}$$

This means $\int_0^\infty e^{-st}\left(\frac{\sin at}{t}\right) dt = \cot^{-1}\frac{s}{a}$

Now put $s = 0$,

$$\therefore \quad \int_0^\infty \frac{\sin at}{t} dt = \cot^{-1}(0) = \frac{\pi}{2}.$$

Ex. 11 : Evaluate $\int_0^\infty t\, e^{-t^2} erf(t)\, dt$.

Sol. : Putting $t^2 = y$, $2t\, dt = dy$.

$$\int_0^\infty t\, e^{-t^2} erf(t)\, dt = \int_0^\infty e^{-y} erf(\sqrt{y}) \frac{dy}{2} \qquad \text{.......... (1)}$$

But $L\left[erf(\sqrt{t})\right] = \frac{1}{s\sqrt{s+1}}$

$$\therefore \quad \int_0^\infty e^{-st} erf(\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$$

Putting $s = 1$, $\int_0^\infty e^{-t}\, erf(\sqrt{t}) = \frac{1}{\sqrt{2}}$

Hence, from (1) $\int_0^\infty t\, e^{-t^2}\, erf(t)\, dt = \frac{1}{2\sqrt{2}}$

EXERCISE

Evaluate the following integrals by using Laplace transform.

1. $\int_0^\infty \frac{e^{-2t} - e^{-3t}}{t} dt$
2. $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$ **(M.U. 1997, 2002, 04)**
3. $\int_0^\infty \frac{\cos 4t - \cos 3t}{t} dt$
4. $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$ **(M.U. 1997, 2005)**
5. $\int_0^\infty e^{-2t} \sin^2 2t\, dt$
6. $\int_0^\infty e^{-t} \sin^2(3t/2)\, dt$
7. $\int_0^\infty \frac{\sin 2t}{t} dt$
8. $\int_0^\infty \frac{e^{-t}\sin t}{t} dt$ **(M.U. 2004)**
9. $\int_0^\infty e^{-t} \sin t \sinh t\, dt$
10. $\int_0^\infty e^{-t}\left(\frac{\sin 3t + \sin 2t}{t}\right) dt$
11. $\int_0^\infty t\, e^{-3t^2} erf(t)\, dt$
12. $\int_0^\infty e^{-2t} \sinh t \cdot \frac{\sin t}{t} dt$ **(M.U. 2002)**
13. $\int_0^\infty e^{-t} \cdot \frac{(1-\cos 2t)}{2t} dt$
14. $\int_0^\infty e^{-t} \cdot \frac{(\cos 3t - \cos 2t)}{t} dt$
15. $\int_0^\infty \frac{e^{-t} \cdot \sin\sqrt{3} \cdot t}{t} dt$ **(M.U. 2003)**

[**Ans. :** **(1)** log (3/2), **(2)** log 3, **(3)** log (3/4), **(4)** log (2/3), **(5)** (1/4) log 5, **(6)** (1/4) log 10, **(7)** $\pi / 2$, **(8)** $\pi / 4$, **(9)** (1/2) $\tan^{-1} 2$, **(10)** $3\pi / 4$, **(11)** 1/8 **(12)** (1/2) $\tan^{-1}$ (1/2), **(13)** (1/4) log 5, **(14)** (1/2) log (1/2), **(15)** $\pi / 3$.]

10. Laplace Transform Of Derivatives

$$\boxed{Lf'(t) = -f(0) + sLf(t)}$$ **(M.U. 2004)** .... (16)

Proof : By definition of Laplace transform,

$$Lf'(t) = \int_0^\infty e^{-st} f'(t)\, dt. \text{ Integrating by parts}$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)\, e^{-st} f(t)\, dt$$

$$\therefore \quad Lf'(t) = -f(0) + s\int_0^\infty e^{-st} f(t)\, dt = -f(0) + sLf(t) \qquad \text{.......... (1)}$$

Applying (1) again,

$$Lf''(t) = -f'(0) + s[Lf'(t)]$$

$$= -f'(0) + s[-f(0) + sLf(t)]$$

$$\therefore \quad Lf''(t) = -f'(0) - sf(0) + s^2 Lf(t) \qquad \text{.......... (2)}$$

Further, $Lf'''(t) = -f''(0) - sf'(0) - s^2 f(0) + s^3 Lf(t)$ .... (3)

In general

$$Lf^n(t) = -f^{n-1}(0) - s \cdot f^{n-2}(0) - s^2 \cdot f^{n-3}(0) + \ldots + s^n Lf(t)$$

Cor. 1 : If $f(0) = f'(0) = f''(0) = \ldots = 0$, we get

$$\boxed{\begin{array}{l} Lf'(t) = s \cdot Lf(t), \quad Lf''(t) = s^2 Lf(t), \\ L^3 f'''(t) = s^3 Lf(t) \ldots\ldots Lf^n(t) = s^n Lf(t) \end{array}} \qquad \text{.......... (4)}$$

Note : These results are going to be highly useful to solve differential equations.

Ex. 1 : Given $f(t) = t + 1,\ 0 \le t \le 2$ and $f(t) = 3, \quad t > 2$, find $L[f(t)]$, $L[f'(t)]$ and $L[f''(t)]$ **(M.U. 2003)**

Sol. : By definition $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$.

$$\therefore \quad L[f(t)] = \int_0^2 e^{-st}(t+1)\, dt + \int_2^\infty e^{-st} \cdot 3\, dt$$

$$= \left[(t+1) \cdot \left(\frac{e^{-st}}{-s}\right) - (1) \cdot \frac{e^{-st}}{s^2}\right]_0^2 + 3\left[\frac{e^{-st}}{-s}\right]_2^\infty = \frac{1}{s} + \frac{1}{s^2}\left(1 - e^{-2s}\right)$$

Now, as shown above,

$$L[f'(t)] = -f(0) + sL[f(t)]$$

But by data $f(0) = 1$,

$$\therefore \quad L[f'(t)] = -1 + s\left[\frac{1}{s} + \frac{1}{s^2}\left(1 - e^{-2s}\right)\right]$$

$$= -1 + 1 + \frac{1}{s}\left(1 - e^{-2s}\right) = \frac{1}{s}\left(1 - e^{-2s}\right)$$

Further, $L[f''(t)] = s^2 L[f(t)] - s[f(0)] - f'(0)$

$$= s^2\left[\frac{1}{s} + \frac{1}{s^2}(1 - e^{-2s}) - s - 1\right]$$

$[\because f(t) = t + 1, f'(t) = 1 \quad \therefore f(0) = 1, f'(0) = 1]$

Ex. 2 : Given $f(t) = \dfrac{\sin t}{t}$, find $L[f'(t)]$. **(M.U. 2002)**

Sol. : $L \sin t = \dfrac{1}{s^2 + 1} \qquad \therefore \quad L\left(\dfrac{\sin t}{t}\right) = \displaystyle\int_s^\infty \dfrac{ds}{s^2 + 1} = \left[\tan^{-1} s\right]_s^\infty$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

But $L[f'(t)] = sLf(t) - f(0)$

$$= s \cot^{-1} s - 1.$$

Ex. 3 : Find Laplace transform of $\dfrac{d}{dt}\left(\dfrac{1 - \cos 2t}{t}\right)$. **(M.U. 2003)**

Sol. : We have $L(1 - \cos 2t) = L(1) - L(\cos 2t)$

$$= \frac{1}{s} - \frac{s}{s^2 + 2^2}$$

$$\therefore \quad L\left(\frac{1 - \cos 2t}{t}\right) = \int_0^\infty \Phi(s)\, ds \qquad \text{[By (15) page 3-36]}$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2 + 2^2}\right] ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2 + 2^2)\right]_s^\infty$$

$$= \left[\log \frac{s}{\sqrt{s^2 + 2^2}}\right] = \left[0 - \log \frac{s}{\sqrt{s^2 + 2^2}}\right]$$

$$= \log\left(\frac{\sqrt{s^2 + 2^2}}{s}\right)$$

But $L[f'(t)] = sL[f(t)] - f(0)$ where $f(t) = \dfrac{1-\cos 2t}{t} = \dfrac{2\sin^2 t}{t}$

$$\therefore \quad f(0) = 2\sin t\left(\frac{\sin t}{t}\right) = 0$$

$$\therefore \quad L\left[\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)\right] = s\log\left(\frac{\sqrt{s^2+2^2}}{s}\right)$$

Alternatively : We may differentiate $\left(\dfrac{1-\cos 2t}{t}\right)$ w.r.t. t and then obtain the Laplace transform of the derivative.

EXERCISE

1. Given $f(t) = 3,\ 0 \le t < 5,\ f(t) = 0,\ t > 5$. Find $L[f(t)]$ and also $L[f'(t)]$.
2. Given $f(t) = t,\ 0 \le t < 3,\ f(t) = 6,\ t > 3$. Find $L[f(t)]$ and also $L[f'(t)]$.
3. Find the Laplace transform of $\dfrac{d}{dt}\left(\dfrac{\sin t}{t}\right)$.

[**Ans.:** (1) $\dfrac{3}{5}\left[1-e^{-5s}\right],\ \dfrac{3s}{5}\left(1-\dfrac{5}{s}-e^{-5s}\right)$

(2) $\dfrac{1}{s^2}+e^{-3s}\left(\dfrac{3}{s}-\dfrac{1}{s^2}\right),\ \dfrac{1}{s}+e^{-3s}\left(3-\dfrac{1}{s}\right)$.

(3) $s\cot^{-1}s - 1$]

11. Laplace Transforms Of Integrals

If $L[f(t)] = \Phi(s)$, then $\boxed{L\int_0^t f(u)\,du = \dfrac{1}{s}\Phi(s)}$ (M.U. 1996) .... (17)

Proof : By definition,

$$L\left[\int_0^t f(u)\,du\right] = \int_0^\infty e^{-st}\left[\int_0^t f(u)\,du\right]dt \qquad \text{.......... (1)}$$

$$= \left[\int_0^t f(u)\,du \cdot \left(\frac{-e^{-st}}{s}\right)\right]_0^\infty - \int_0^\infty\left[\left(\frac{-e^{-st}}{s}\right)\frac{d}{dt}\int_0^t f(u)\,du\right]$$

(By integration by parts)

But $\dfrac{d}{dt}\displaystyle\int_0^t f(u)\,du = f(t)$

$$\therefore \quad L\left[\int_0^t f(u)\,du\right] = \int_0^\infty \frac{1}{s}\cdot e^{-st} f(t)\,dt = \frac{1}{s}\cdot L[f(t)] = \frac{1}{s}\Phi(s)$$

Since $\Phi(s) = L[f(t)]$, we have

$$L\int_0^t f(u)\,du = \frac{1}{s}L\{f(t)\}$$

Corollary : The above result can be generalised as follows.

$$\boxed{L\left[\int_0^t\int_0^t \cdots\cdots \int_0^t f(u)(du)^n\right] = \frac{1}{s^n}L[f(t)]}$$

Ex. 1 : Find the Laplace transform of

(i) $\displaystyle\int_0^t \sin 2u\,du$ (ii) $\displaystyle\int_0^t u\cosh u \cdot du$. **(M.U. 1998, 99)**

Sol. : (i) Since $L\sin 2t = \dfrac{2}{s^2+4} = \Phi(s)$, say

$$\therefore \quad L\int_0^t \sin 2u\,du = \frac{1}{s}\Phi(s) = \frac{2}{s(s^2+4)}$$

(ii) Now, $L\cos ht = \dfrac{s}{s^2-a^2}$ $\quad\therefore\quad Lt\cos ht = \dfrac{d}{ds}\left[\dfrac{s}{s^2-a^2}\right]$ (by § 8)

$$\therefore \quad Lt\cos ht = -\frac{s^2+a^2}{(s^2-a^2)^2} = \Phi(s), \text{ say}$$

$$\therefore \quad L\int_0^t u\cos hu\,du = \frac{1}{s}\Phi(s) = -\frac{s^2+a^2}{s(s^2-a^2)^2}$$

Ex. 2 : Find the Laplace transform of the following.

(i) $erf\sqrt{t}$ **(M.U. 1996, 97, 2003)**

(ii) $erf\sqrt{2t}$ and evaluate $\displaystyle\int_0^\infty erf(2\sqrt{t})\cdot e^{-5t}\,dt$. **(M.U. 2005)**

(iii) $e^{3t}erf\sqrt{t}$ **(M.U. 1998)** (iv) $e^{-3t}erf\sqrt{t}$ **(M.U. 1997, 99)**

Sol. : (i) We have $erf\,x = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-u^2}du$

Hence, $erf\sqrt{t} = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^{\sqrt{t}} e^{-u^2}du$

Now put $u^2 = v$ $\quad\therefore\quad u = \sqrt{v}$ $\quad\therefore\quad du = \dfrac{1}{2\sqrt{v}}dv$

When $u = 0,\ v = 0$; When $u = \sqrt{t},\ v = t$

$$\therefore \quad erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^t e^{-v}\cdot\frac{1}{2\sqrt{v}}dv = \frac{1}{\sqrt{\pi}}\int_0^t e^{-v}v^{-1/2}dv$$

By Ex. 9 page 3-5.

$$L(v^{-1/2}) = \frac{\overline{|1/2}}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\therefore \quad L\left(e^{-v}\, v^{-1/2}\right) = \frac{\sqrt{\pi}}{\sqrt{s+1}} \qquad \text{(By first shifting theorem)}$$

$$\therefore \quad L\int_0^t e^{-v}\, v^{-1/2} dv = \frac{\sqrt{\pi}}{s\sqrt{s+1}}$$

$$\therefore \quad L\, erf\sqrt{t} = \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{s\sqrt{s+1}} = \frac{1}{s\sqrt{s+1}}.$$

(See also Ex. 3 page 3-12)

(ii) Now $erf\, 2\sqrt{t} = erf\sqrt{4t}$

But $erf\sqrt{t} = \dfrac{1}{s\sqrt{s+1}}$

Hence, by § 5 page 3-16.

$$erf\sqrt{4t} = \frac{1}{4}\, \frac{1}{(s/4)\sqrt{(s/4)+1}} = \frac{2}{s\sqrt{s+4}}$$

$$\therefore \quad erf\, 2\sqrt{t} = \frac{2}{s\sqrt{s+4}}$$

By definition of Laplace transform

$$\int_0^\infty e^{-st}\, erf\, 2\sqrt{t} = \frac{2}{s\sqrt{5+4}}$$

$$\therefore \quad \int_0^\infty e^{-5t}\, erf\, 2\sqrt{t} = \frac{2}{5\sqrt{5+4}} = \frac{2}{15}.$$

(iii) Since, $L(erf\sqrt{t}) = \dfrac{1}{s\sqrt{s+1}}$

$$L\left(e^{3t} erf\sqrt{t}\right) = \frac{1}{(s-3)\sqrt{s-3+1}} \qquad \text{[By § 6, page 3-18]}$$

$$= \frac{1}{(s-3)\sqrt{s-2}}$$

(iv) Similarly, $L\left(e^{-3t} erf\sqrt{t}\right) = \dfrac{1}{(s+3)\sqrt{s+3+1}} = \dfrac{1}{(s+3)\sqrt{s+4}}$

Ex. 3 : Find $\int_0^\infty e^{-t} erf\sqrt{t}\, dt.$ **(M.U. 1997)**

Sol. : As proved above $L[erf\sqrt{t}] = \dfrac{1}{s\sqrt{s+1}}$

This means $\int_0^\infty e^{-st} \cdot erf\sqrt{t}\, dt = \dfrac{1}{s\sqrt{s+1}}$

Now put $s = 1$

$$\therefore \quad \int_0^\infty e^{-t} \cdot erf\sqrt{t}\, dt = \frac{1}{\sqrt{2}}$$

Ex. 4 : Find $L\, e^{3t} \cdot t\, erf\sqrt{t}.$ **(M.U. 2000)**

Sol. :
$$L\, t\, erf\sqrt{t} = -\frac{d}{ds}\left(\frac{1}{s\sqrt{s+1}}\right)$$

$$= -\left[-\frac{1}{s^2(s+1)}\left\{\frac{1}{2\sqrt{s+1}} + \sqrt{s+1}\right\}\right]$$

$$= \frac{1}{s^2(s+1)} \cdot \frac{s+2(s+1)}{2\sqrt{s+1}} = \frac{3s+2}{2s^2(s+1)^{3/2}}$$

$$\therefore \quad L\, e^{3t} \cdot t \cdot erf\sqrt{t} = \frac{3(s-3)+2}{2(s-3)^2(s-3+1)^{3/2}}$$

$$= \frac{3s-7}{2(s-3)^2(s-2)^{3/2}}.$$

Ex. 5 : Find $L[erf_c\sqrt{t}]$.

Sol. : As derived above, $L[erf\sqrt{t}] = \dfrac{1}{s\sqrt{s+1}}.$

Since, $erf\, x + erf_c x = 1,$ we get

$$erf\sqrt{t} + erf_c\sqrt{t} = 1 \quad \therefore \quad erf_c\sqrt{t} = 1 - erf\sqrt{t}$$

$$\therefore \quad L[erf_c\sqrt{t}] = L(1) - L[erf\sqrt{t}]$$

$$= \frac{1}{s} - \frac{1}{s\sqrt{s+1}} = \frac{\sqrt{s+1}-1}{s\sqrt{s+1}}$$

$$= \frac{\sqrt{s+1}-1}{s\sqrt{s+1}} \cdot \frac{\sqrt{s+1}+1}{\sqrt{s+1}+1} = \frac{s+1-1}{s\sqrt{s+1}(\sqrt{s+1}+1)}$$

$$= \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$$

Ex. 6 : Find $\int_0^\infty e^{-t} erf_c\sqrt{t}\, dt.$ **(M.U. 1996)**

Sol. : As proved above $L[erf_c\sqrt{t}] = \dfrac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}.$

By definition of Laplace transform this means

$$\int_0^\infty e^{-st} erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{s+1}\,(\sqrt{s+1}+1)}$$

Put $s = 1$, we get $\int_0^\infty e^{-t} erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{2}\,(\sqrt{2}+1)} = \frac{1}{\sqrt{6}}$

Ex. 7 : Find $\int_0^\infty erf_c\sqrt{t}\, dt$.

Sol. : As proved above $L[erf_c\sqrt{t}] = \frac{1}{\sqrt{s+1}\,(\sqrt{s+1}+1)}$.

This means $\int_0^\infty e^{-st} erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{s+1}\,(\sqrt{s+1}+1)}$

Putting $s = 0$, we get $\int_0^\infty erf_c\sqrt{t}\, dt = \frac{1}{2}$.

Ex. 8 : Find $L\left[\int_0^t \int_0^t \int_0^t t\sin t\,(dt)^3\right]$. **(M.U. 2005)**

Sol. : By the corollary

$$L\left[\int_0^t \int_0^t \int_0^t t\sin t\,(dt)^3\right] = \frac{1}{s^3} L[t\sin t]$$

But by Ex. (2) (i), page 3-27,

$$L[t\sin t] = \frac{2s}{(s^2+1)^2}$$

$$\therefore\ L\left[\int_0^t \int_0^t \int_0^t t\sin t\,(dt)^3\right] = \frac{1}{s^3}\cdot\frac{2s}{(s^2+1)^2} = \frac{2}{s^2(s^2+1)^2}.$$

Ex. 9 : Find the Laplace transform of the following.

(i) $\int_0^t u\cos^2 u\, du$. **(M.U. 1995)** **(ii)** $\int_0^t u\, e^{-3u}\cos^2 2u\, du$. **(M.U. 1993)**

(iii) $\int_0^t u^{-1} e^{-u}\sin u\, du$. **(M.U. 1996)** **(iv)** $\int_0^t \frac{1-e^{-au}}{u}\, du$. **(M.U. 1997, 2003)**

(v) $\int_0^t \frac{\sin u}{u}\, du$. **(M.U. 1997, 2002)** **(vi)** $\int_t^\infty \frac{\cos u}{u}\, du$. **(M.U. 1997)**

(vii) $\int_0^t u^2 \sin u\, du$. **(M.U. 2003)**

Sol. : (i) $\cos^2 u = \frac{1+\cos 2u}{2}$

$$L\cos^2 u = L\frac{1}{2}(1+\cos 2u) = \frac{1}{2}L(1) + \frac{1}{2}L\cos 2u$$

$$= \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+2^2}$$

$$L(u\cos^2 u) = -\frac{d}{ds}\cdot\left(\frac{1}{2}\cdot\frac{1}{s}\right) - \frac{d}{ds}\cdot\frac{1}{2}\cdot\left(\frac{s}{s^2+2^2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{2}\frac{(s^2+2^2)\cdot 1 - s\cdot 2s}{(s^2+2^2)^2} \quad \text{[By § 8 page 3-25]}$$

$$= \frac{1}{2s^2} + \frac{1}{2}\cdot\frac{s^2-2^2}{(s^2+2^2)^2} = \Phi(s), \text{ say}$$

$$\therefore\ L\int_0^t u\cos^2 u\, du = \frac{1}{s}\cdot\Phi(s) \quad \text{[By § 11 page 3-52]}$$

$$= \frac{1}{2s^3} + \frac{1}{2}\cdot\frac{s^2-2^2}{s(s^2+2^2)^2}$$

(ii) $\cos^2 2u = \frac{1+\cos 4u}{2}$

$$L\cos^2 2u = L\frac{1}{2}(1+\cos 4u) = \frac{1}{2}L(1) + \frac{1}{2}L(\cos 4u)$$

$$= \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+4^2}$$

$$L e^{-3u}\cos^2 2u = \frac{1}{2}\cdot\frac{1}{s+3} + \frac{1}{2}\cdot\frac{s+3}{(s+3)^2+4^2}$$

$$= \frac{1}{2}\cdot\frac{1}{s+3} + \frac{1}{2}\cdot\frac{s+3}{s^2+6s+25}$$

$$L u\, e^{-3u}\cos^2 2u = -\frac{d}{ds}\left[\frac{1}{2}\cdot\frac{1}{s+3}\right] - \frac{d}{ds}\left[\frac{1}{2}\cdot\frac{s+3}{s^2+6s+25}\right]$$

[By (14) page 3-25]

$$= \frac{1}{2}\cdot\frac{1}{(s+3)^2} - \frac{1}{2}\left[\frac{(s^2+6s+25)-(s+3)(2s+6)}{(s^2+6s+25)^2}\right]$$

$$= \frac{1}{2(s+3)^2} + \frac{1}{2}\cdot\frac{s^2+6s-7}{(s^2+6s+25)^2} = \Phi(s), \text{ say}$$

$$\therefore\ L\int_0^t u\, e^{-3u}\cos^2 2u\, du = \frac{1}{s}\Phi(s) \quad \text{[By § 11 page 3-52]}$$

$$= \frac{1}{2s(s+3)^2} + \frac{1}{2}\cdot\frac{s^2+6s-7}{s(s^2+6s+25)^2}$$

(iii) $L\sin u = \frac{1}{s^2+1}$; $L\,e^{-u}\sin u = \frac{1}{(s+1)^2+1}$. [By shifting theorem]

$$\therefore\ L\left(\frac{1}{u}e^{-u}\sin u\right) = \int_s^{\infty}\frac{ds}{(s+1)^2+1} \qquad \text{[By (15) page 3-36]}$$

$$= \left[\tan^{-1}(s+1)\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1) = \Phi(s), \text{ say}$$

$$\therefore\ L\int_0^t \frac{e^{-u}\sin u}{u}\,du = \frac{1}{s}\Phi(s) = \frac{1}{s}\cot^{-1}(s+1).$$

(iv) $L(1-e^{au}) = L(1) - L(e^{au}) = \frac{1}{s} - \frac{1}{s-a}$

$$\therefore\ L\left(\frac{1-e^{au}}{u}\right) = \int_s^{\infty}\left[\frac{1}{s} - \frac{1}{s-a}\right]ds$$

$$= \left[\log\frac{s}{s-a}\right]_s^{\infty} = \left[0 - \log\frac{s}{s-a}\right] = \log\left(\frac{s-a}{s}\right)$$

$$\therefore\ L\int_0^t\left(\frac{1-e^{-u}}{u}\right)du = \frac{1}{s}\Phi(s) = \frac{1}{s}\log\left(\frac{s-a}{s}\right).$$

(v) $L\sin u = \frac{1}{s^2+1}$

$$\frac{L\sin u}{u} = \int_s^{\infty}\Phi(s)\,ds = \int_s^{\infty}\frac{ds}{s^2+1}$$

$$= \left[\tan^{-1}s\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\therefore\ L\int_0^t \frac{\sin u}{u}\,du = \frac{1}{s}\Phi(s) = \frac{1}{s}\cot^{-1}s.$$

(vi) Let $f(t) = \int_t^{\infty}\frac{\cos u}{u}\,du$

Now put $u = vt$, $du = t\,dv$

When $u = t$, $v = 1$; when $u \to \infty$, $v \to \infty$

$$\therefore\ f(t) = \int_1^{\infty}\frac{\cos vt}{vt}\cdot t\,dv = \int_1^{\infty}\frac{\cos vt}{v}\,dv$$

$$\therefore\ L[f(t)] = \int_0^{\infty}e^{-st}\left[\int_0^{\infty}\frac{\cos vt}{v}\,dv\right]dt$$

$$= \int_1^{\infty}\frac{dv}{v}\int_0^{\infty}e^{-st}\cos vt\,dt = \int_1^{\infty}\frac{dv}{v}L[\cos vt]\,dt$$

$$= \int_1^{\infty}\frac{s\,dv}{v(s^2+v^2)} = \frac{1}{s}\int_1^{\infty}\left[\frac{1}{v} - \frac{v}{s^2+v^2}\right]dv$$

$$= \frac{1}{s}\left[\log v - \frac{1}{2}\log(s^2+v^2)\right]_1^{\infty} = \frac{1}{2s}\left[\log\frac{v^2}{s^2+v^2}\right]_1^{\infty}$$

$$= \frac{1}{2s}\left[0 - \log\frac{1}{s^2+1}\right] = \frac{1}{2s}\log(s^2+1)$$

Ex. 10 : Find the Laplace transforms of the following.

(i) $t\int_0^t e^{-4u}\sin 3u\,du$ **(ii)** $e^{-t}\int_0^t \frac{\sin u}{u}\,du$

(iii) $t^{-1}\int_0^t e^{-u}\sin u\,du$ **(iv)** $e^{-4t}\int_0^t u\sin 3u\,du$ **(M.U. 2003, 04)**

(v) $\cos ht\int_0^t e^{u}\cos hu\,du$ **(M.U. 1996)**

Sol. : (i) We have $L(\sin 3u) = \frac{3}{s^2+9}$ [By (7), page 3-5]

$$\therefore\ L(e^{-4u}\sin 3u) = \frac{3}{(s+4)^2+9} \qquad \text{[By (12), page 3-18]}$$

$$\therefore\ L\left[\int_0^t e^{-4u}\sin 3u\,du\right] = \frac{1}{s}\cdot\frac{3}{(s+4)^2+9} \qquad \text{[By (17), page 3-52]}$$

$$\therefore\ L\left[t\int_0^t e^{-4u}\sin 3u\,du\right] = (-1)\frac{d}{ds}\left[\frac{3}{s^3+8s^2+25s}\right]$$

[By (14), page 3-25]

$$= \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

(ii) As proved in (v) of the above Ex. 8,

$$L\left[\int_0^t\frac{\sin u}{u}\,du\right] = \frac{1}{s}\cdot\cot^{-1}s$$

$$\therefore\ L\left[e^{-t}\int_0^t\frac{\sin u}{u}\,du\right] = \frac{1}{(s+1)}\cot(s+1) \qquad \text{[By (12), page 3-18]}$$

(iii) We have $L(\sin u) = \frac{1}{s^2+1}$ [By (7), page 3-5]

$$L(e^{-u}\sin u) = \frac{1}{(s+1)^2+1} \qquad \text{[By (12), page 3-18]}$$

$$\therefore\ L\left[\int_0^t e^{-u}\sin u\,du\right] = \frac{1}{s}\cdot\frac{1}{s^2+2s+2} \qquad \text{[By (17), page 3-52]}$$

$$\therefore\ L\left[\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right] = \int_s^\infty \frac{ds}{s(s^2+2s+2)} \quad \text{[By (15), page 3-36]}$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s+2}{s^2+2s+2}\right] ds \quad \text{[By partial fractions]}$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{1}{2}\cdot\frac{2(s+2)}{(s^2+2s+2)}\right] ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{1}{2}\cdot\frac{2s+2}{(s^2+2s+2)} - \frac{2}{2(s^2+2s+2)}\right] ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{1}{2}\cdot\frac{2s+2}{s^2+2s+2} - \frac{1}{(s+1)^2+1}\right] ds$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+2s+2) - \tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+2s+2}} - \tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\left(\log 1 - \frac{\pi}{2}\right) - \left(\log\frac{s}{\sqrt{2s^2+2s+2}} - \tan^{-1}(s+1)\right)\right]$$

$$= \frac{1}{2}\left[\log\frac{\sqrt{s^2+2s+2}}{s} - \left(\frac{\pi}{2} - \tan^{-1}(s+1)\right)\right]$$

$$= \left[\frac{1}{4}\log\left(\frac{s^2+2s+2}{s^2}\right) - \frac{1}{2}\cot^{-1}(s+1)\right]$$

(iv) We have $L(\sin 3u) = \dfrac{3}{s^2+9}$ [By (7), page 3-5]

$$\therefore\ L(u\sin 3u) = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) \quad \text{[By (14), page 3-25]}$$

$$= \frac{3\cdot 2s}{(s^2+9)^2}$$

$$\therefore\ L\left[\int_0^t u\sin 3u\,du\right] = \frac{1}{s}\cdot\frac{6s}{(s^2+9)^2} = \frac{s}{(s^2+9)^2} \quad \text{[By (17), page 3-52]}$$

$$\therefore\ L\left[e^{-4t}\int_0^t u\sin 3u\,du\right] = \frac{6}{[(s+4)^2+9]^2} \quad \text{[By (12), page 3-18]}$$

$$= \frac{6}{(s^2+8s+25)^2}$$

(v) We have $L(\cos hu) = \dfrac{s}{s^2-1}$ [By (8), page 3-5]

$$\therefore\ L(e^u\cos hu) = \frac{s-1}{(s-1)^2-1} \quad \text{[By (12), page 3-18]}$$

$$= \frac{s-1}{s^2-2s+1-1} = \frac{s-1}{s(s-2)}$$

$$\therefore\ L\left[\int_0^t e^u\cos h\,u\ du\right] = \frac{1}{s}\cdot\frac{s-1}{s(s-2)} \quad \text{[By (17), page 3-52]}$$

$$= \frac{s-1}{s^2(s-2)}$$

$$\therefore\ L\left[\cos hu\cdot\int_0^t e^u\cos hu\,du\right] = L\left[\frac{e^u+e^{-u}}{2}\int_0^t e^u\cos hu\,du\right]$$

$$= \frac{1}{2}\left[L\left\{e^u\int_0^t e^u\cos hu\,du\right\} + L\left\{e^{-u}\int_0^t e^u\cos hu\,du\right\}\right]$$

$$= \frac{1}{2}\left[\frac{(s-1)-1}{(s-1)^2(s-1-2)} + \frac{(s+1)-1}{(s+1)^2(s+1-2)}\right]$$

[By (12), page 3-18]

$$= \frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)} + \frac{s}{(s+1)^2(s-1)}\right]$$

Note : Note that in all these examples we have used more than one rules to find the Laplace transform. We start with the Laplace transform of the innermost function and arrive at the Laplace transform of the given function step by step.

EXERICSE

(A) Find the Laplace transform of

1. $\int_0^t ue^{-3u}\sin 4u\,du$ **(M.U. 2002)**
2. $\int_0^t \frac{1}{u}\sin 3u\,du$
3. $\int_0^t e^{-3u}\sin 4u\,du$
4. $\int_0^t \frac{1-e^{-au}}{u}du$ **(M.U. 2003)**

[**Ans. :** (1) $\dfrac{8(s+3)}{s(s^2+6s+25)^2}$, (2) $\dfrac{1}{s}\cot^{-1}s$, (3) $\dfrac{1}{s}\cdot\dfrac{4}{s^2+6s+25}$, (4) $\dfrac{1}{s}\log\left(\dfrac{s+a}{s}\right)$]

(B) Find the Laplace transform of

1. $\int_0^t e^{-u} u^4 du$ 2. $\int_0^t e^{-u} \cos u\, du$ 3. $\int_0^t \frac{1+e^{-u}}{u} du$

4. $\int_0^t \frac{e^u \sin u}{u} du$ **(M.U. 1997)** 5. $e^{-3u} \int_0^t u \sin 3u\, du$

6. $\int_0^t e^u \cdot \frac{\sin 4u}{u} du$ **(M.U. 2000)** 7. $\int_0^t u \cdot e^{-2u} \sin 3u\, du$ **(M.U. 2002)**

8. $\int_0^t u \cdot e^{-3u} \sin^2 u\, du$ **(M.U. 2004)**

[**Ans. :** (1) $\frac{4!}{s(s+1)^5}$, (2) $\frac{1}{s} \cdot \frac{s+1}{(s^2+2s+2)}$, (3) $\frac{1}{s} \log[s(s+1)]$,

(4) $\frac{1}{s} \cot^{-1}(s-1)$, (5) $-\frac{6}{(s^2+6s+18)^2}$, (6) $\frac{1}{s} \cdot \cot^{-1}\left(\frac{s+1}{4}\right)$,

(7) $\frac{1}{s} \cdot \frac{3(2s+4)}{(s^2+4s+13)^2}$, (8) $\frac{1}{2s}\left[\frac{1}{(s+3)^2} + \frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$

(C) Find the following.

1. $\int_0^\infty e^{-8t} erf\sqrt{t}\, dt$ 2. $\int_0^\infty e^{-3t} erf_c\sqrt{t}\, dt$ [**Ans. :** (1) 1/24, (2) 1/6]

(D) Find 1. $L\left[\int_0^t \int_0^t t \cdot \sin t\, (dt)^2\right]$, 2. $L\left[\int_0^t \int_0^t \int_0^t \cos 2t\, (dt)^3\right]$.

[**Ans. :** (1) $\frac{1}{s(s^2+1)^2}$, (2) $\frac{1}{s^2(s^2+4)}$.]

(a) Particular Value of the Function $L\left[\int_0^t f(t)\, dt\right]$

As in the earlier cases, § 4(a), page 3-14; § 8(a), page 3-33; § 9(a), page 3-42, we can find the value of certain definite integrals, using the Laplace transform. This is illustrated in the following examples.

Ex. : Evaluate the following integral by using Laplace transforms.

(i) $\int_0^\infty e^{-2t}\left(\int_0^t \frac{e^{-u} \sin u}{u} du\right) dt$

(ii) $\int_0^\infty e^{-t}\left(\int_0^t u^2 \sin hu \cos hu\, du\right) dt$ **(M.U. 2003)**

Sol. : (i) As seen in Ex. 8(ii) above

$$L\left[\int_0^t \frac{e^{-u} \sin u}{u} du\right] = \frac{1}{s} \cot^{-1}(s+1)$$

By definition of Laplace transform this means

$$\int_0^\infty e^{-st}\left[\int_0^t \frac{e^{-u} \sin u}{u} du\right] dt = \frac{1}{s} \cdot \cot^{-1}(s+1)$$

Putting $s = 2$, we get

$$\int_0^\infty e^{-2t}\left[\int_0^t \frac{e^{-u} \sin u}{u} du\right] dt = \frac{1}{2} \cot^{-1}(3).$$

(ii) We have $L(\sin hu \cos hu) = L\left(\frac{1}{2} \sin h 2u\right)$

$$= \frac{1}{2} \cdot \frac{2}{s^2+2^2} = \frac{1}{s^2+4}$$

$$\therefore \quad L(u^2 \sin hu \cos hu) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s^2+4}\right)$$

$$= \frac{d}{ds}\left(-\frac{2s}{(s^2+4)^2}\right) = -2\left[\frac{(s^2+4)^2 - s \cdot 2(s^2+4) \cdot 2s}{(s^2+4)^4}\right]$$

$$= -2\left[\frac{s^2+4-4s^2}{(s^2+4)^3}\right] = 2 \cdot \frac{(3s^2-4)}{(s^2+4)^3} = \Phi(s) \text{ say}$$

$$\therefore \quad L\left[\int_0^t u^2 \sin hu \cos hu\, du\right] = \frac{1}{s} \cdot \Phi(s)$$

$$= \frac{2}{s} \cdot \frac{(3s^2-4)}{(s^2+4)^3} \quad \text{[By (17), page 3-52]}$$

By definition of Laplace transform, this means,

$$\int_0^\infty e^{-st}\left[\int_0^t u^2 \sin hu \cos hu\, du\right] dt = \frac{2}{s} \frac{(3s^2-4)}{(s^2+4)^3}$$

Putting $s = 1$, we get

$$\int_0^\infty e^{-t}\left[\int_0^t u^2 \sin hu \cos hu\, du\right] dt = \frac{2}{1} \cdot \frac{(3 \cdot 1 - 4)}{(1+4)^3} = -\frac{2}{125}$$

EXERCISE

Evaluate the following integrals by using Laplace transforms.

1. $\int_s^\infty e^{-t}\left(\int_0^t u \cos^2 u\, du\right) dt$ 2. $\int_0^\infty e^{-2t}\left[\int_0^t\left(\frac{1-e^{-u}}{u}\right) du\right] dt$

3. $\int_0^\infty e^{-4t}\left(\cos h\, t \int_0^t e^u \cos hu\, du\right) dt$

4. $\int_0^\infty e^{-t}\left(t\int_0^t e^{-4u}\cos u\,du\right)dt$ **5.** $\int_0^\infty e^{-t}\left(\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right)dt$

[**Ans.: (1)** 12 / 50, **(2)** (1 / 2) log (3 / 2), **(3)** 31 / 225,
(4) 9 / 64, **(5)** (1 / 4) log 5 – (1 / 2) $\cot^{-1}$ (2)]

Theory

1. Define Laplace transform of $f(t)$, $t > 0$. **(M.U. 1995, 2002)**
Also state the conditions for its existence. **(M.U. 2005)**

2. Define Laplace transform of a function of t and state the rule of change of scale with one example. **(M.U. 2002, 04)**

3. State and prove first shifting theorem. Hence, find $L\{e^{2t}\cos t\cos 2t\}$.

(M.U. 2003) [**Ans.:** $\dfrac{(s-2)(s^2-4s+9)}{(s^2-4s+13)(s^2-4s+5)}$]

4. If $L[f(t)] = \Phi(s)$, prove that $L\left[e^{-at}f(t)\right] = \Phi(s+a)$. **(M.U. 1995)**

5. If
$$g(t) = \begin{cases} f(t-a) & t > a \\ 0 & t < a \end{cases}$$
and $L[f(t)] = \Phi(s)$, prove that $L[g(t)] = e^{-as}\Phi(s)$. **(M.U. 1995)**

6. If $L[f(t)] = \Phi(s)$, prove that $L\left[t^n f(t)\right] = (-1)^n \dfrac{d^n}{ds^n}\Phi(s)$.
(M.U. 1997, 2005)

7. State and prove First Shifting Theorem. **(M.U. 1997, 99, 2003)**

8. If $L[f(t)] = \Phi(s)$, prove that $L[f(at)] = \dfrac{1}{a}\Phi\left(\dfrac{s}{a}\right)$.

9. If $L[f(t)] = \Phi(s)$, prove that $L[t\,f(t)] = -\Phi'(s)$. **(M.U. 1993)**

10. If $L[f(t)] = \Phi(s)$, prove that $L\left[\dfrac{1}{t}f(t)\right] = \int_s^\infty \Phi(s)\,ds$.
(M.U. 1994, 98, 2000)

11. Prove that $L[f'(t)] = -f(0) + s\,L\,f(t)$.

12. If $L[f(t)] = \Phi(s)$, prove that $L\int_0^t f(u)\,du = \dfrac{1}{s}\Phi(s)$. **(M.U. 1996)**

13. State and prove second shifting theorem. **(M.U. 2002)**

14. If $J_0(t) = 1 - \dfrac{t^2}{2^2} + \dfrac{t^4}{2^2\cdot 4^2} - \dfrac{t^6}{2^2\cdot 4^2\cdot 6^2} + \ldots,$
prove that $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$.

❀ ❀ ❀

6

LAPLACE TRANSFORMS - II

1. Introduction

In this chapter we shall study the methods of obtaining inverse Laplace transform. We shall further study Laplace transforms of some special functions. And then we shall apply the Laplace transforms to solve differential equations and boundary value problems.

2. Inverse Laplace Transforms

Definition : If $L[f(t)] = \Phi(s) = \int_0^\infty e^{-st} f(t)\,dt$ then $f(t)$ is called the **inverse Laplace transform** of $\Phi(s)$ and can be denoted as $L^{-1}\Phi(s) = f(t)$.

We get the following inverse transforms of some standard functions from this definition.

Table of Inverse Transforms

(i) $L(1) = \dfrac{1}{s}$ $\therefore$ $\boxed{L^{-1}\left(\dfrac{1}{s}\right) = 1}$ .... (1)

(ii) $L(e^{-at}) = \dfrac{1}{s+a}$ $\therefore$ $\boxed{L^{-1}\left(\dfrac{1}{s+a}\right) = e^{-at}}$ .... (2)

(iii) $L(e^{at}) = \dfrac{1}{s-a}$ $\therefore$ $\boxed{L^{-1}\left(\dfrac{1}{s-a}\right) = e^{at}}$ .... (3)

(iv) $L(t^{n-1}) = \dfrac{\lfloor n}{s^n}$ $\therefore$ $\boxed{L^{-1}\left(\dfrac{1}{s^n}\right) = \dfrac{t^{n-1}}{\lfloor n}}$ .... (4)

(v) $L(t^{n-1}) = \dfrac{(n-1)!}{s^n}$ $\therefore$ $\boxed{L^{-1}\left(\dfrac{1}{s^n}\right) = \dfrac{t^{n-1}}{(n-1)!}}$ .... (5)

if n is a positive integer.

(vi) $L(\sin at) = \dfrac{a}{s^2+a^2}$ $\therefore$ $\boxed{L^{-1}\left(\dfrac{1}{s^2+a^2}\right) = \dfrac{1}{a}\sin at}$ .... (6)

(vii) $L(\cos at) = \dfrac{s}{s^2 + a^2}$ $\therefore$ $\boxed{L^{-1}\left(\dfrac{s}{s^2 + a^2}\right) = \cos at}$ .... (7)

(viii) $L(\sin h\, at) = \dfrac{a}{s^2 - a^2}$ $\therefore$ $\boxed{L^{-1}\left(\dfrac{1}{s^2 - a^2}\right) = \dfrac{1}{a}\sin h\, at}$ .... (8)

(ix) $L(\cos h\, at) = \dfrac{s}{s^2 - a^2}$ $\therefore$ $\boxed{L^{-1}\left(\dfrac{s}{s^2 - a^2}\right) = \cos h\, at}$ .... (9)

EXERCISE

Write down the inverse Laplace transforms of the following.

1. $\dfrac{1}{s^5}$, 2. $\dfrac{1}{s^{3/2}}$, 3. $\dfrac{1}{s^{1/2}}$, 4. $\dfrac{1}{s^6}$, 5. $\dfrac{1}{s^2+9}$, 6. $\dfrac{1}{s^2+25}$,

7. $\dfrac{1}{s-4}$, 8. $\dfrac{1}{s+4}$, 9. $\dfrac{s}{s^2+4}$, 10. $\dfrac{s}{s^2+16}$, 11. $\dfrac{1}{s^2-1}$,

12. $\dfrac{1}{s^2-4}$, 13. $\dfrac{s}{s^2-4}$, 14. $\dfrac{s}{s^2-36}$, 15. $\dfrac{1}{s}$.

[**Ans. :** Not given for obvious reason.]

Ex. 1 : If $L[f(t)] = \dfrac{s+2}{s^2+2}$, find $L[f'(t)]$. **(M.U. 2003)**

Sol. : We have obtained $L[f'(t)]$ and $L[f''(t)]$ on page 3-50, when $f(t)$ was given as a function of t. Here, we are given $L[f(t)]$ in terms of s.

$$\because\ L[f(t)] = \frac{s+2}{s^2+2} \qquad \therefore\ f(t) = L^{-1}\left[\frac{s+2}{s^2+2}\right]$$

$$\therefore\ f(t) = L^{-1}\left[\frac{s}{s^2+2}\right] + L^{-1}\left[\frac{2}{s^2+2}\right]$$

$$= \cos\sqrt{2}\,t + 2\cdot\frac{1}{\sqrt{2}}\sin\sqrt{2}\cdot t$$

Putting $t = 0$, $f(0) = \cos 0 = 1$.

Now, $L[f'(t)] = -f(0) + s\,L[f(t)]$ [By 16 page 3-50]

$$\therefore\ L[f'(t)] = -1 + s\cdot\frac{s+2}{s^2+2} = \frac{s^2+2s}{s^2+2} - 1$$

$$= \frac{2s-2}{s^2+2} = \frac{2(s-1)}{s^2+2}$$

Ex. 2 : If $L[f(t)] = \dfrac{s+3}{s^2+4}$, find $L[f'(t)]$.

Sol. : Do it yourself. [**Ans. :** $\dfrac{3s-4}{s^2+4}$]

Ex. 3 : Prove that $L^{-1}\left[\dfrac{1}{s}\cos\dfrac{1}{s}\right] = 1 - \dfrac{t^2}{(2!)^2} + \dfrac{t^4}{(4!)^2} - \dfrac{t^6}{(6!)^2} + ...$

Sol. : We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

$$\therefore\ \cos\frac{1}{s} = 1 - \frac{1}{2!}\cdot\frac{1}{s^2} + \frac{1}{4!}\cdot\frac{1}{s^4} - \frac{1}{6!}\cdot\frac{1}{s^6} + ...$$

$$\therefore\ \frac{1}{s}\cos\frac{1}{s} = \frac{1}{s} - \frac{1}{2!}\cdot\frac{1}{s^3} + \frac{1}{4!}\cdot\frac{1}{s^5} - \frac{1}{6!}\cdot\frac{1}{s^7} + ...$$

$$\therefore\ L^{-1}\left[\frac{1}{s}\cos\frac{1}{s}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{2!}\cdot\frac{1}{s^3} + \frac{1}{4!}\cdot\frac{1}{5^5} - \frac{1}{6}\cdot\frac{1}{s^7} + ...\right]$$

$$= 1 - \frac{1}{2!}\cdot\frac{t^2}{2!} + \frac{1}{4!}\cdot\frac{t^4}{4!} - \frac{1}{6!}\cdot\frac{t^6}{6!}$$

$$= 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + ...$$

Miscellaneous Examples

Ex. 1 : Find $\int_0^\infty \cos(tx^2)\,dx$ and hence, find $\int_0^\infty \cos x^2 \cdot dx$.

Sol. : Let $f(t) = \int_0^\infty \cos(tx^2)\,dx$

$$\therefore\ L f(t) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\int_0^\infty [\cos(tx^2)\,dx]\,dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st}\cos(tx^2)\,dt\right]dx$$

$$= \int_0^\infty [L\cos(tx^2)]\,dx = \int_0^\infty \frac{s}{s^2+x^4}\,dx$$

Now put $x = \sqrt{s\tan\theta}$ $\therefore$ $dx = \dfrac{s\cdot\sec^2\theta\,d\theta}{2\sqrt{s\tan\theta}}$

$$\therefore\ L[f(t)] = \int_0^{\pi/2}\frac{s}{s^2+s^2\tan^2\theta}\cdot\frac{s\cdot\sec^2\theta\,d\theta}{2\sqrt{s\tan\theta}}$$

$$= \int_0^{\pi/2}\frac{1}{2\sqrt{s\tan\theta}}\,d\theta = \frac{1}{2\sqrt{s}}\int_0^{\pi/2}(\sin\theta)^{-1/2}(\cos\theta)^{1/2}\,d\theta$$

$$= \frac{1}{2\sqrt{s}} \frac{\overline{|1/4}\ \overline{|3/4}}{2 \cdot 1!} \left[\int_0^{\pi/2} \sin^p \theta \cos^p \theta \, d\theta = \frac{\overline{|(p+1)/2}\ \overline{|(q+1)/2}}{2\overline{|(p+q+2)/2}} \right]$$

$$= \frac{1}{2\sqrt{s}} \cdot \frac{\sqrt{2} \cdot \pi}{2} \qquad \left[\because \overline{|1/4}\ \overline{|3/4} = \sqrt{2} \cdot \pi \right]$$

$$\therefore \quad L(f(t)) = \frac{\pi}{2\sqrt{2}\sqrt{s}}$$

$$\therefore \quad f(t) = \frac{\pi}{2\sqrt{2}} L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{\pi}{2\sqrt{2}} \cdot \frac{t^{-1/2}}{\overline{|1/2}} = \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{1}{\sqrt{t}}$$

Now put $t = 1$, $\quad \therefore \int_0^\infty \cos x^2 dx = \dfrac{\sqrt{\pi}}{2\sqrt{2}} = \dfrac{1}{2}\sqrt{\dfrac{\pi}{2}}$.

Ex. 2 : Find $\int_0^\infty \sin(tx^2)\, dx$ and hence, find $\int_0^\infty \sin x^2 \, dx$. **(M.U. 2003)**

Sol. : Let $f(t) = \int_0^\infty \sin(tx^2) \, dx$

$$\therefore \quad L f(t) = \int_0^\infty e^{-st} f(t) \, dt = \int_0^\infty e^{-st} \left[\int_0^\infty \sin(tx^2) \, dx \right] dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} \sin(tx^2) \, dt \right] dx$$

$$= \int_0^\infty [L \sin(tx^2)] \, dx = \int_0^\infty \frac{x^2}{s^2 + x^4} dx$$

Now put $x = \sqrt{s \tan\theta}$, $dx = \dfrac{s \cdot \sec^2\theta \, d\theta}{2\sqrt{s\tan\theta}}$

$$\therefore \quad L f(t) = \int_0^{\pi/2} \frac{s\tan\theta}{s^2 + s^2\tan^2\theta} \cdot \frac{s \cdot \sec^2\theta}{2\sqrt{s\tan\theta}} \cdot d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta = \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$$

$$= \frac{1}{2\sqrt{s}} \frac{\overline{|3/4}\ \overline{|1/4}}{2 \cdot 1} = \frac{1}{2\sqrt{s}} \frac{\sqrt{2} \cdot \pi}{2}$$

$$= \frac{\pi}{2\sqrt{2}} \cdot \frac{1}{\sqrt{s}}$$

$$\therefore \quad f(t) = \frac{\pi}{2\sqrt{2}} L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{\pi}{2\sqrt{2}} \cdot \frac{t^{-1/2}}{\overline{|1/2}} = \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{1}{\sqrt{t}}$$

Now put $t = 1$, $\quad \therefore \int_0^\infty \sin x^2 dx = \dfrac{\sqrt{\pi}}{2\sqrt{2}} = \dfrac{1}{2}\sqrt{\dfrac{\pi}{2}}$.

Ex. 3 : Find $\int_0^\infty e^{-tx^2} dx$ and hence, find $\int_0^\infty e^{-x^2} dx$. **(M.U. 2004)**

Sol. : Let $f(t) = \int_0^\infty e^{-tx^2} dx$

$$\therefore \quad L f(t) = \int_0^\infty e^{-st} \left[\int_0^\infty e^{-tx^2} dx \right] dt = \int_0^\infty \left[\int_0^\infty e^{-st} \cdot e^{-tx^2} dt \right] dx$$

$$= \int_0^\infty \left[L(e^{-tx^2}) \right] dx = \int_0^\infty \frac{dx}{s + x^2} \qquad \left[\because L e^{-at} = \frac{1}{s+a} \right]$$

$$= \left[\frac{1}{\sqrt{s}} \tan^{-1} \frac{x}{\sqrt{s}} \right]_0^\infty = \frac{\pi}{2\sqrt{s}}$$

$$\therefore \quad f(t) = \frac{\pi}{2} L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{\pi}{2} \frac{t^{-1/2}}{\overline{|1/2}}$$

$$= \frac{\pi}{2} \frac{1}{\sqrt{\pi}\sqrt{t}} = \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

Now, put $t = 1$, $\quad \therefore \int_0^\infty e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$.

3. Methods Of Obtaining Inverse Laplace Transforms

There are various methods of obtaining inverse Laplace transform. The choice of the method depends upon the nature of the problem. Sometimes an example can be solved by more than one method. You will find below that some solved problems are repeated but note that they are solved by different methods. You are advised to find which method you find more easy.

(a) Use of standard results : Using the above result inverse Laplace transforms of some simple functions can be obtained as illustrated below :

Ex. : Find the inverse Laplace transform of

(i) $\dfrac{2}{s} + \dfrac{1}{s^3} + \dfrac{1}{s+4}$, (ii) $\dfrac{3s+4}{s^2+16}$.

Sol. : (i) From the above table, we get,

$$L^{-1}\left[\frac{2}{s} + \frac{1}{s^3} + \frac{1}{s+4}\right] = 2L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s+4}\right)$$

$$= 2 + \frac{t^2}{2} + e^{-4t}$$

(ii) $$L^{-1}\left(\frac{3s+4}{s^2+16}\right) = 3L^{-1}\left(\frac{s}{s^2+4^2}\right) + L^{-1}\left(\frac{4}{s^2+4^2}\right)$$

$$= 3 \cdot \cos 4t + \sin 4t$$

EXERCISE

Find the inverse Laplace transform of

1. $\frac{3+2s+s^2}{s^3}$, 2. $\frac{2s+3}{s^2+9}$, 3. $\frac{3s+5}{9s^2-25}$, 4. $\frac{1}{4s+5}$,

5. $\frac{1}{4s-5}$, 6. $\frac{4s+15}{16s^2-25}$, 7. $\frac{(s^2-1)^2}{s^5}$, 8. $\frac{s+3}{s^2+4}$

[**Ans. :** (1) $3\cdot\frac{t^2}{2}+2\cdot t+1$, (2) $2\cos 3t+\sin 3t$, (3) $\frac{1}{3}e^{(5/3)t}$,

(4) $\frac{1}{4}e^{-(5/4)t}$, (5) $\frac{1}{4}e^{(5/4)t}$, (6) $\frac{1}{4}\cos h\left[\frac{5}{4}\cdot t\right]+\frac{3}{4}\sin h\frac{5}{4}\cdot t$,

(7) $1-t^2+\frac{t^4}{24}$, (8) $\cos 2t+\frac{3}{2}\sin 2t$.]

(b) Using shifting theorem : We know that,

if $L[f(t)]=\Phi(s)$, then $Le^{-at}f(t)=\Phi(s+a)$.

This means if $f(t)=L^{-1}\Phi(s)$ then $L^{-1}[\Phi(s+a)]=e^{-at}f(t)$.

i.e. $\quad \mathbf{L^{-1}[\Phi(s+a)]=e^{-at}L^{-1}\Phi(s)}$

e.g. $\quad L^{-1}\frac{1}{(s+2)^2}=e^{-2t}\cdot L^{-1}\frac{1}{s^2}=e^{-2t}\cdot t$

$$L^{-1}\left[\frac{(s+1)}{(s+1)^2+3^2}\right]=e^{-t}L^{-1}\frac{s}{s^2+3^2}=e^{-t}\cdot\cos 3t$$

Note the following results

(i) $L^{-1}\frac{1}{(s-b)^n}=e^{bt}L^{-1}\frac{1}{s^n}=e^{bt}\cdot\frac{t^{n-1}}{\lfloor n}$ .... (10)

(ii) $L^{-1}\frac{1}{(s-b)^n}=e^{bt}L^{-1}\frac{1}{s^n}=e^{bt}\cdot\frac{t^{n-1}}{(n-1)!}$ .... (11)

(if n is an integer)

(iii) $L^{-1}\frac{1}{(s-b)^2+a^2}=e^{bt}L^{-1}\frac{1}{s^2+a^2}=\frac{e^{bt}}{a}\sin at$ .... (12)

(iv) $L^{-1}\frac{(s-b)}{(s-b)^2+a^2}=e^{bt}L^{-1}\frac{s}{s^2+a^2}=e^{bt}\cos at$ .... (13)

(v) $L^{-1}\frac{1}{(s-b)^2-a^2}=e^{bt}L^{-1}\frac{1}{s^2-a^2}=\frac{e^{bt}}{a}\sin h\,at$ .... (14)

(vi) $\boxed{L^{-1}\frac{(s-b)}{(s-b)^2-a^2}=e^{bt}L^{-1}\frac{s}{s^2-a^2}=e^{bt}\cos h\,at}$ .... (15)

EXERCISE

Write down the inverse Laplace transform of the following.

1. $\frac{1}{s-2}$, 2. $\frac{1}{s+3}$, 3. $\frac{1}{(s-2)^{3/2}}$, 4. $\frac{1}{(s+3)^2}$, 5. $\frac{1}{(s-2)^3}$,

6. $\frac{1}{(s+5)^{5/2}}$, 7. $\frac{1}{(s-2)^2+2^2}$, 8. $\frac{(s-3)}{(s-3)^2+2^2}$, 9. $\frac{1}{(s-3)^2+4^2}$,

10. $\frac{(s-4)}{(s-4)^2+5^2}$, 11. $\frac{1}{(s-2)^2-2^2}$, 12. $\frac{(s-3)}{(s-3)^2-2^2}$,

13. $\frac{1}{(s-3)^2-4^2}$, 14. $\frac{(s-4)}{(s-4)^2-5^2}$.

[**Ans. :** Answers not given for obvious reason.]

Ex. 1 : Find the inverse Laplace transform of $\frac{s}{(s-2)^6}$.

Sol. : $L^{-1}\frac{s}{(s-2)^6}=L^{-1}\left[\frac{(s-2)+2}{(s-2)^6}\right]=L^{-1}\left[\frac{1}{(s-2)^5}+\frac{2}{(s-2)^6}\right]$

$$=L^{-1}\frac{1}{(s-2)^5}+2L^{-1}\frac{1}{(s-2)^6}=e^{2t}L^{-1}\frac{1}{s^5}+2e^{2t}L^{-1}\frac{1}{s^6}$$

$$=e^{2t}\left[\frac{t^4}{4!}\right]+2e^{2t}\left[\frac{t^5}{5!}\right]=e^{2t}\left[\frac{t^4}{4!}+2\frac{t^5}{5!}\right].$$

Ex. 2 : Find the inverse Laplace transform of

(i) $\frac{s+2}{s^2-4s+13}$ (ii) $\frac{4s+12}{s^2+8s+12}$ **(M.U. 2003)**

Sol. : (i) $L^{-1}\left[\frac{s+2}{s^2-4s+13}\right]=L^{-1}\left[\frac{s+2}{(s-2)^2+3^2}\right]$

$$=L^{-1}\left[\frac{(s-2)+4}{(s-2)^2+3^2}\right]=e^{2t}L^{-1}\left[\frac{s+4}{s^2+3^2}\right]$$

$$=e^{2t}L^{-1}\frac{s}{s^2+3^2}+4\cdot e^{2t}L^{-1}\left(\frac{1}{s^2+3^2}\right)$$

$$=e^{2t}\cdot\cos 3t+\frac{4}{3}e^{2t}\sin 3t$$

(ii) $L^{-1}\left[\dfrac{4s+12}{s^2+8s+12}\right] = L^{-1}\left[\dfrac{4(s+4)-2^2}{(s+4)^2-2^2}\right]$

$$= L^{-1}\left[\frac{4(s+4)}{(s+4)^2-2^2}\right] - L^{-1}\left[\frac{2^2}{(s+4)^2-2^2}\right]$$

$$= 4e^{-4t}L^{-1}\left[\frac{s}{s^2-2^2}\right] - 4e^{-4t}L^{-1}\left[\frac{1}{s^2-2^2}\right]$$

$$= 4e^{-4t}\cos h\,2t - 4e^{-4t}\cdot\frac{1}{4}\sin h\,2t$$

$$= e^{-4t}(4\cos h\,2t - \sin h\,2t)$$

Ex. 3 : Find : (i) $L^{-1}\left[\dfrac{s+2}{s^2+4s+7}\right]$ **(M.U. 1993),**

(ii) $L^{-1}\left[\dfrac{2s+3}{s^2+2s+2}\right]$, **(M.U. 1994)** (iii) $L^{-1}\left[\dfrac{3s+7}{s^2-2s-3}\right]$. **(M.U. 1999, 2003)**

Sol. : (i) $L^{-1}\left[\dfrac{s+2}{s^2+4s+7}\right] = L^{-1}\left[\dfrac{(s+2)}{(s+2)^2+(\sqrt{3})^2}\right]$

$$= e^{-2t}L^{-1}\frac{s}{s^2+(\sqrt{3})^2} = e^{-2t}\cdot\cos\sqrt{3}\,t.$$

(ii) $L^{-1}\left[\dfrac{2s+3}{s^2+2s+2}\right] = L^{-1}\left[\dfrac{2(s+1)+1}{(s+1)^2+1}\right]$

$$= 2L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= 2e^{-t}L^{-1}\left[\frac{s}{s^2+1}\right] + e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2e^{-t}\cos t + e^{-t}\cdot\sin t = e^{-t}[2\cos t + \sin t].$$

(iii) $L^{-1}\left[\dfrac{3s+7}{s^2-2s-3}\right] = L^{-1}\left[\dfrac{3(s-1)+10}{(s-1)^2-2^2}\right]$

$$= 3L^{-1}\left[\frac{s-1}{(s-1)^2-(2)^2}\right] + 10\,L^{-1}\frac{1}{(s-1)^2-(2)^2}$$

$$= 3e^{t}L^{-1}\left[\frac{s}{s^2-(2)^2}\right] + 10e^{t}L^{-1}\left[\frac{1}{s^2-(2)^2}\right]$$

$$= 3e^{t}\cos h\,2t + \frac{10}{2}e^{t}\sin h\,2t$$

$$= e^{t}(3\cos h\,2t + 5\sin h\,2t).$$

EXERCISE

Find the inverse Laplace transform of

1. $\dfrac{6s-4}{s^2-4s+20}$ 2. $\dfrac{3s-7}{s^2-6s+8}$, 3. $\dfrac{1}{(s-3)^3}$,

4. $\dfrac{1}{(s+2)^4}$, 5. $\dfrac{1}{(s+1)^2}+\dfrac{s-2}{s^2-4s+5}+\dfrac{s-2}{s^3-4s+3}$,

6. $\dfrac{s}{s^2+2s+2}$ **(M.U. 2004)** 7. $\dfrac{s}{(2s\pm1)^2}$ **(M.U. 2003)**

[**Ans. : (1)** $6\cdot e^{2t}\cos 4t + 2e^{2t}\sin 4t$, **(2)** $3\cdot e^{3t}\cdot\cos ht + 2\cdot e^{3t}\sin ht$,

(3) $\dfrac{1}{2}e^{3t}t^2$, **(4)** $\dfrac{1}{3!}e^{-2t}\cdot t^3$, **(5)** $e^{-t}\cdot t + e^{2t}\cos t + e^{2t}\cos ht$,

(6) $e^{-t}(\cos t - \sin t)$, **(7)** $e^{-t/2}(t-4)/16$.]

(c) Method of partial fractions : Whenever possible we express the given function $\Phi(s)$ into the sum of linear or quadratic partial fractions as $\Phi(s) = \dfrac{A}{(s+a)^r} + \dfrac{Bs+C}{(s^2+a^2)^r}$ and then use the standard results given above to find L^{-1}.

Ex. 1 : Find inverse Laplace transform of

(i) $\dfrac{s^2+16s-24}{s^4+20s^2+64}$ (ii) $\dfrac{s+29}{(s+4)(s^2+9)}$ **(M.U. 1999)**

(iii) $\dfrac{s+2}{(s+3)(s+1)^3}$ (iv) $\dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$ **(M.U. 2005)**

(v) $\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}$ (vi) $\dfrac{s+2}{(s^2+4s+8)(s^2+4s+13)}$.

Sol. : (i) $\Phi(s) = \dfrac{s^2+16s-24}{s^4+20s^2+64} = \dfrac{s^2+16s-24}{(s^2+4)(s^2+16)}$

Let $\Phi(s) = \dfrac{as+b}{s^2+4} + \dfrac{cs+d}{s^2+16}$

$$\therefore\quad s^2+16s-24 = (as+b)(s^2+16)+(cs+d)(s^2+4)$$

$$= (a+c)s^3+(b+d)s^2+(16a+4c)s+(16b+4d)$$

Equating the coefficients of like powers of s we get, $a+c=0$, $b+d=1$, $16a+4c=16$, $16b+4d=-24$.

$\therefore\quad a = 4/3,\ c = -4/3,\ b = -7/3,\ d = 10/3$

$$\therefore \quad \Phi(s) = \frac{1}{3}\cdot\frac{4s-7}{s^2+4} - \frac{1}{3}\cdot\frac{4s-10}{s^2+16}$$

$$\therefore \quad L^{-1}[\Phi(s)] = \frac{4}{3}L^{-1}\left(\frac{s}{s^2+4}\right) - \frac{7}{3}L^{-1}\left(\frac{1}{s^2+4}\right) - \frac{4}{3}L^{-1}\left(\frac{s}{s^2+16}\right) + \frac{10}{3}L^{-1}\left(\frac{1}{s^2+16}\right)$$

$$= \frac{4}{3}\cos 2t - \frac{7}{3}\cdot\frac{1}{2}\sin 2t - \frac{4}{3}\cdot\cos 4t + \frac{10}{3}\cdot\frac{1}{4}\sin 4t$$

(ii) $\Phi(s) = \dfrac{s+29}{(s+4)(s^2+9)} = \dfrac{a}{s+4} + \dfrac{bs+c}{s^2+9}$, say

$$\therefore \quad s+29 = (a+b)s^2 + (c+4b)s + (9a+4c)$$

Equating the coefficient of like powers of s, we get,

$$\therefore \quad a+b=0,\ c+4b=1,\ 9a+4c=29$$

$$\therefore \quad a=1,\ b=-1,\ c=5.$$

$$\therefore \quad \Phi(s) = \frac{1}{s+4} - \frac{s-5}{s^2+9}$$

$$\therefore \quad L^{-1}[\Phi(s)] = L^{-1}\left(\frac{1}{s+4}\right) - L^{-1}\left(\frac{s}{s^2+9}\right) + 5L^{-1}\left(\frac{1}{s^2+9}\right)$$

$$= e^{-4t} - \cos 3t + 5\cdot\frac{1}{3}\sin 3t$$

(iii) $\Phi(s) = \dfrac{s+2}{(s+3)(s+1)^3}$

$$= \frac{1}{8}\cdot\frac{1}{(s+3)} - \frac{1}{8}\cdot\frac{1}{(s+1)} + \frac{1}{4}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{(s+1)^3}$$

(after expressing the function in partial fractions)

$$\therefore \quad L^{-1}[\Phi(s)] = \frac{1}{8}\cdot L^{-1}\left(\frac{1}{s+3}\right) - \frac{1}{8}\cdot L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{4}\cdot L^{-1}\frac{1}{(s+1)^2} + \frac{1}{2}\cdot L^{-1}\frac{1}{(s+1)^3}$$

But $L^{-1}\left[\dfrac{1}{(s+b)^{n+1}}\right] = \dfrac{1}{n!}\cdot e^{-bt}\cdot t^n$

$$\therefore \quad L^{-1}[\Phi(s)] = \frac{1}{8}\cdot e^{-3t} - \frac{1}{8}\cdot e^{-t} + \frac{1}{4}\cdot\frac{1}{1!}e^{-t}\cdot t + \frac{1}{2}\cdot\frac{1}{2!}e^{-t}\cdot t^2$$

$$= \frac{1}{8}\left[(2t^2+2t-1)e^{-t} + e^{-3t}\right]$$

(iv) $\Phi(s) = \dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$

$$= \frac{as+b}{(s^2+2s+5)} + \frac{cs+d}{(s^2+2s+2)}$$

Simplifying and equating like powers of s on both sides, we get, $a=0$, $b=3$, $c=0$, $d=-2$

$$\therefore \quad \Phi(s) = \frac{3}{s^2+2s+5} - \frac{2}{s^2+2s+2}$$

$$= \frac{3}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+1^2}$$

$$\therefore \quad L^{-1}\Phi(s) = 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right] - 2\cdot L^{-1}\left[\frac{1}{(s+1)^2+1^2}\right]$$

$$= 3e^{-t}L^{-1}\left[\frac{1}{s^2+2^2}\right] - 2e^{-t}L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$= 3e^{-t}\cdot\frac{1}{2}\sin 2t - 2e^{t}\cdot 1\cdot\sin t$$

(v) Let us first consider

$$\frac{1}{(s^2+1)(s^2+4)(s^2+9)} = \frac{1/24}{s^2+1} - \frac{1/15}{s^2+4} + \frac{1/40}{s^2+9} \qquad \textbf{(Note this)}$$

Now, $\dfrac{1}{(s^2+1)(s^2+4)(s^2+9)} = \dfrac{1}{24}\cdot\dfrac{s}{s^2+1} - \dfrac{1}{15}\cdot\dfrac{s}{s^2+4} + \dfrac{1}{40}\cdot\dfrac{s}{s^2+9}$

$$\therefore \quad L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right]$$

$$= \frac{1}{24}L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{15}L^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{40}L^{-1}\left(\frac{s}{s^2+9}\right)$$

$$= \frac{1}{24}\cos t - \frac{1}{15}\cos 2t + \frac{1}{40}\cos 3t$$

(vi) $\dfrac{s+2}{(s^2+4s+8)(s^2+4s+13)} = \dfrac{s+2}{[(s+2)^2+2^2][(s+2)^2+3^2]}$

$$\therefore \quad L^{-1}\frac{s+2}{(s^2+4s+8)(s^2+4s+13)}$$

$$= L^{-1}\frac{1}{5}\left[\frac{s+2}{(s+2)^2+2^2} - \frac{s+2}{(s+2)^2+3^2}\right] \qquad \text{(As above)}$$

$$= \frac{1}{5}\left[e^{-2t}L^{-1}\frac{s}{s^2+2^2} - e^{-2t}L^{-1}\frac{s}{s^2+3^2}\right]$$

$$= \frac{e^{-2t}}{5}(\cos 2t - \cos 3t)$$

Ex. 2 : Find the inverse Laplace transform of the following.

(i) $\dfrac{3s+1}{(s+1)(s^2+2)}$ **(M.U. 1995),** (ii) $\dfrac{5s^2-15s-11}{(s+1)(s-2)^2}$ **(M.U. 1995),**

(iii) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$ **(M.U. 1996, 98, 2000)** (iv) $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ **(M.U. 2004)**

(v) $\dfrac{5s^2+8s-1}{(s+3)(s^2+1)}$ **(M.U. 1996, 2002)** (vi) $\dfrac{2s}{s^4+4}$ **(M.U. 1997, 2002)**

(vii) $\dfrac{s+2}{s^2(s+3)}$ **(M.U. 1997, 2003)** (viii) $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$ **(M.U. 1996, 2003, 04, 05)**

Sol. : (i) Let $\dfrac{3s+1}{(s+1)(s^2+2)} = \dfrac{a}{s+1} + \dfrac{bs+c}{s^2+2}$

$\therefore \; 3s+1 = a(s^2+2)+(s+1)(bs+c)$

$= (a+b)s^2+(b+c)s+2a+c.$

Equating the coefficients of like powers of s

$\therefore \; a+b=0,\; b+c=3,\; 2a+c=1 \quad \therefore \; a=-\dfrac{1}{2},\; b=\dfrac{1}{2},\; c=\dfrac{5}{2}$

$$\therefore \; L^{-1}\left[\frac{3s+1}{(s+1)(s^2+2)}\right] = -\frac{1}{2}L^{-1}\frac{1}{s+1}+\frac{1}{2}L^{-1}\frac{s}{s^2+2}+\frac{5}{2}L^{-1}\frac{1}{s^2+2}$$

$$= -e^{-t}L^{-1}\frac{1}{s}+\frac{1}{2}L^{-1}\frac{s}{s^2+(\sqrt{2})^2}+\frac{5}{2}L^{-1}\frac{1}{s^2+(\sqrt{2})^2}$$

$$= -e^{-t}\cdot 1+\frac{1}{2}\cos\sqrt{2}\,t+\frac{5}{2\sqrt{2}}\sin\sqrt{2}\,t.$$

(ii) Let $\dfrac{5s^2-15s-11}{(s+1)(s-2)^2} = \dfrac{a}{s+1}+\dfrac{b}{s-2}+\dfrac{c}{(s-2)^2}$

$\therefore \; 5s^2-15s-11 = a(s-2)^2+b(s+1)(s-2)+c(s+1)$

Equating the coefficient of like power of s

$a+b=5,\; -4a-b-c=-15,\; 4a-2b+c=-11$

$\therefore \; a=1,\; b=4,\; c=-7.$

$$\therefore \; L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^2}\right] = L^{-1}\frac{1}{s+1}+4L^{-1}\frac{1}{s-2}-7L^{-1}\frac{1}{(s-2)^2}$$

$$= e^{-t}L^{-1}\frac{1}{s}+4e^{2t}L^{-1}\frac{1}{s}-7e^{2t}L^{-1}\frac{1}{s^2}$$

$$= e^{-t}+4e^{2t}-7e^{2t}t.$$

(iii) Let $s^2 = x \quad \therefore \; \dfrac{s^2}{(s^2+a^2)(s^2+b^2)} = \dfrac{x}{(x+a^2)(x+b^2)}$

Let $\dfrac{x}{(x+a^2)(x+b^2)} = \dfrac{A}{x+a^2}+\dfrac{B}{x+b^2}$

$\therefore \; x = A(x+b^2)+B(x+a^2)$

Put $x=-a^2 \quad \therefore \; -a^2 = A(-a+B^2) \quad \therefore \; A = a^2/(a^2-b^2)$

Put $x=-b^2 \quad \therefore \; -b^2 = B(-b^2+a^2) \quad \therefore \; B = -b^2/(a^2-b^2)$

$$\therefore \; \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2-b^2}\left[\frac{a^2}{s^2+a^2}-\frac{b^2}{s^2+b^2}\right].$$

$$\therefore \; L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{1}{a^2-b^2}\left[L^{-1}\left(\frac{a^2}{s^2+a^2}\right)-L^{-1}\left(\frac{b^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[a^2\cdot L^{-1}\left(\frac{1}{s^2+a^2}\right)-b^2\cdot L^{-1}\left(\frac{1}{s^2+b^2}\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[\frac{a^2}{a}\cdot\sin at-b^2\cdot\frac{1}{b}\sin bt\right]$$

$$= \frac{1}{a^2-b^2}(a\sin at-b\sin bt)$$

[For another method see Ex. 2 (ii) page 4-21]

(iv) Let us first consider

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{1}{s+a^2}-\frac{1}{s^2+b^2}\right] \quad \textbf{(Note this)}$$

$$\therefore \; L^{-1}\frac{s}{(s^2+a^2)(s^2+b^2)} = L^{-1}\frac{1}{b^2-a^2}\left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right]$$

$$= \frac{1}{(b^2-a^2)}\left[L^{-1}\frac{s}{s^2+a^2}-L^{-1}\frac{s}{s^2+b^2}\right]$$

$$= \frac{1}{b^2-a^2}(\cos at-\cos bt)$$

[For another method see Ex. 3 (v) page 4-24]

(v) Let $\dfrac{5s^2+8s-1}{(s+3)(s^2+1)} = \dfrac{a}{s+3} + \dfrac{bs+c}{s^2+1}$

$\therefore \quad 5s^2+8s-1 = a(s^2+1) + (bs+c)(s+3)$

$= (a+b)s^2 + (3b+c)s + (a+3c)$

Equating the coefficients of like powers of s, we get,

$a+b=5, \; 3b+c=8, \; a+3c=-1$

$\therefore \quad a=2, \; b=3, \; c=-1$

$$\therefore \quad L^{-1}\left[\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right] = L^{-1}\left[\frac{2}{s+3} + 3\cdot\frac{s}{s^2+1} - \frac{1}{s^2+1}\right]$$

$$= 2L^{-1}\frac{1}{s+3} + 3L^{-1}\frac{s}{s^2+1} - L^{-1}\frac{1}{s^2+1}$$

$$= 2e^{-3t} + 3\cos t - \sin t$$

(vi) $$\frac{2s}{s^4+4} = \frac{2s}{(s^4+4s^2+4)-4s^2}$$

$$= \frac{2s}{(s^2+2)^2-(2s)^2} = \frac{1}{2}\left[\frac{1}{(s^2-2s+2)} - \frac{1}{(s^2+2s+2)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore \quad L^{-1}\frac{2s}{s^4+4} = \frac{1}{2}\left[L^{-1}\frac{1}{(s-1)^2+1} - L^{-1}\frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\left[e^{t}L^{-1}\frac{1}{s^2+1} - e^{-t}L^{-1}\frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}\left[e^{t}\sin t - e^{-t}\sin t\right]$$

$$= \sin t\left(\frac{e^{t}-e^{-t}}{2}\right) = \sin t \sinh t.$$

(vii) Let $\dfrac{s+2}{s^2(s+3)} = \dfrac{a}{s} + \dfrac{b}{s^2} + \dfrac{c}{s+3}$

$\therefore \quad s+2 = as(s+3) + b(s+3) + cs^2$

Putting $s=0$, $2=3b$; putting $s=-3$, $-1=9c$.

Equating the coefficients of s^2, $a+c=0$

$\therefore a = 1/9$.

$$\therefore \quad L^{-1}\frac{s+2}{s^2(s+3)} = \frac{1}{9}L^{-1}\left(\frac{1}{s}\right) + \frac{2}{3}L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{9}L^{-1}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{9}(1) + \frac{2}{3}t - \frac{1}{9}e^{-3t} = \frac{1}{9}(1+6t-e^{-3t})$$

(viii) $$\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{(s+1)^2+2}{\left[(s+1)^2+2^2\right]\left[(s+1)^2+1^2\right]}$$

$$\therefore \quad L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = e^{-t}L^{-1}\frac{s^2+2}{(s^2+4)(s^2+1)}$$

Let $s^2 = x$ and $\dfrac{x+2}{(x+4)(x+1)} = \dfrac{a}{x+4} + \dfrac{b}{x+1}$

$\therefore \quad x+2 = a(x+1) + b(x+4)$

When $x=-1$, $1=3b$; when $x=-4$, $-2=-3a$.

$$\therefore \quad \frac{s^2+2}{(s^2+4)(s^2+1)} = \frac{2}{3}\cdot\frac{1}{s^2+4} + \frac{1}{3}\cdot\frac{1}{s^2+1}$$

$$\therefore \quad L^{-1}\frac{s^2+2}{(s^2+4)(s^2+1)} = \frac{2}{3}L^{-1}\frac{1}{s^2+4} + \frac{1}{3}L^{-1}\frac{1}{s^2+1}$$

$$= \frac{2}{3}\cdot\frac{1}{2}\sin 2t + \frac{1}{3}\sin t$$

$$\therefore \quad L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = \frac{e^{-t}}{3}(\sin 2t + \sin t)$$

(For another method see Ex. 2 (iii), page 4-21.)

EXERCISE

Find the inverse Laplace transform of

1. $\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}$ 2. $\dfrac{s+4}{(s^2-1)(s+1)}$ **(M.U. 2003)** 3. $\dfrac{s^2+1}{s^3+3s^2+2s}$

4. $\dfrac{s}{(s^2+5s+6)}$ 5. $\dfrac{3s+7}{s^2-2s-3}$ **(M.U. 2002)** 6. $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$

7. $\dfrac{s^2+10s+13}{(s-1)(s^2-5s+6)}$ 8. $\dfrac{1}{s^2(s+1)}$ 9. $\dfrac{1}{s^2(s+2)}$

10. $\dfrac{s^2}{(s+4)^3}$ 11. $\dfrac{s^2}{(s+1)^3}$ 12. $\dfrac{1}{s^2(s+3)^2}$

13. $\dfrac{1}{(s-2)(s+2)^2}$ 14. $\dfrac{4s+4}{(s-1)^2(s+2)}$ 15. $\dfrac{2s+3}{(s+1)^2(s+2)}$

16. $\dfrac{1}{(s-2)^4(s+3)}$ 17. $\dfrac{1}{s(s^2-a^2)}$ 18. $\dfrac{2s^2-1}{(s^2+1)(s^2+4)}$

19. $\dfrac{s}{(s^2+1)(s^2+4)}$ 20. $\dfrac{s+29}{(s+4)(s^2+9)}$ 21. $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

22. $\dfrac{11s^2-2s+5}{2s^3-3s^2-3s+2}$ 23. $\dfrac{s}{(s-3)(s^2+4)}$ 24. $\dfrac{s^2}{(s^2+1)(s^2+4)}$

25. $\dfrac{2s-1}{(s^4+s^2+1)}$ 26. $\dfrac{s^2}{(s^2+9)(s^2+4)}$ 27. $\dfrac{1}{(s-2)^2(s+3)}$ **(M.U. 2003)**

28. $\dfrac{1}{s^3+1}$ **(M.U. 2003)** 29. $\dfrac{1}{s^3(s-1)}$ **(M.U. 2003)** 30. $\dfrac{s}{(s+1)^2(s^2+1)}$ **(M.U. 2004)**

[**Ans. : (1)** $-\dfrac{1}{6}e^{-t}-\dfrac{4}{3}e^{2t}+\dfrac{7}{2}e^{3t}$, **(2)** $\dfrac{5}{4}e^{t}-\dfrac{5}{4}e^{-t}+\dfrac{3}{2}te^{-t}$

(3) $\dfrac{1}{2}-2e^{-t}+\dfrac{5}{2}e^{-2t}$, **(4)** $3e^{-3t}-2e^{-2t}$,

(5) $4e^{3t}-e^{-t}$, **(6)** $\dfrac{1}{2}e^{t}-e^{2t}+\dfrac{5}{2}e^{3t}$,

(7) $12e^{t}-37e^{2t}+26e^{3t}$, **(8)** $-1+t+e^{-t}$,

(9) $-\dfrac{1}{4}+\dfrac{1}{2}t+\dfrac{1}{4}e^{-2t}$, **(10)** $e^{-4t}(1-8t+8t^2)$,

(11) $e^{-t}\left(1-2t+\dfrac{t^2}{2}\right)$, **(12)** $\dfrac{1}{27}(-2+3t+2e^{-3t}+3t^2e^{-3t})$,

(13) $\dfrac{1}{16}(e^{2t}-e^{-2t}-4te^{-2t})$, **(14)** $\dfrac{4}{9}(e^{t}-e^{-2t}+6te^{t})$,

(15) $e^{-t}-e^{-2t}+te^{-t}$,

(16) $\dfrac{e^{2t}}{6}\left[\dfrac{t^3}{5}-\dfrac{3}{25}t^2+\dfrac{6}{125}t-\dfrac{6}{625}\right]+\dfrac{1}{625}e^{-3t}$,

(17) $\dfrac{1}{2a^2}[e^{at}+e^{-at}-2]$, **(18)** $\dfrac{3}{2}\cdot\sin 2t-\sin t$,

(19) $\dfrac{1}{3}(\cos t-\cos 2t)$, **(20)** $e^{-4t}-\cos 3t+\dfrac{5}{3}\sin 3t$,

(21) $e^{t}-e^{-t}\cos 2t+\dfrac{3}{2}e^{-t}\sin 2t$, **(22)** $2e^{-t}+5e^{2t}-\dfrac{3}{2}e^{t/2}$,

(23) $\dfrac{3}{13}e^{3t}-\dfrac{3}{13}\cos 2t+\dfrac{2}{13}\sin 2t$, **(24)** $-\dfrac{1}{3}\sin t+\dfrac{2}{3}\sin 2t$,

(25) $\dfrac{1}{2}e^{t/2}\cos\dfrac{\sqrt{3}}{2}t+\dfrac{\sqrt{3}}{2}e^{t/2}\sin\dfrac{\sqrt{3}}{2}t-\dfrac{1}{2}e^{-t/2}\cos\dfrac{\sqrt{3}}{2}t-\dfrac{5}{2\sqrt{3}}e^{-t/2}\sin\dfrac{\sqrt{3}}{2}t$, **(26)** $\dfrac{1}{5}(3\sin 3t-2\sin 2t)$,

(27) $-\dfrac{1}{25}e^{2t}+\dfrac{t}{5}e^{2t}+\dfrac{1}{25}e^{-3t}$,

(28) $\dfrac{1}{3}e^{-t}-\dfrac{e^{t/2}}{3}\cdot\cos\left(\dfrac{\sqrt{3}}{2}\cdot t\right)+\dfrac{e^{t/2}}{\sqrt{3}}\cdot\sin\left(\dfrac{\sqrt{3}}{2}\cdot t\right)$,

(29) $1-t+\dfrac{t^2}{2}-e^{-t}$, **(30)** $\dfrac{1}{2}\left[\sin t-t\cdot e^{-t}\right]$.

(d) Using Change of Scale Property : We know that by the change of scale property, if $L[f(t)]=\Phi(s)$ then $L[f(at)]=\dfrac{1}{a}\Phi\left(\dfrac{s}{a}\right)$.

Taking inverse Laplace transforms, this means

$$\boxed{\textbf{if } L^{-1}[\Phi(s)]=f(t) \textbf{ then } L^{-1}\left[\frac{1}{a}\Phi\left(\frac{s}{a}\right)\right]=f(at)}$$

Ex. 1 : If $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}=\dfrac{t}{2}\sin t$, find $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$.

Sol. : By the above rule, since $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}=\dfrac{t}{2}\sin t$ replacing s by $\dfrac{s}{a}$ and t by at (after dividing the given function by a)

$$L^{-1}\left\{\frac{1}{a}\cdot\frac{s/a}{[(s/a)^2+1]^2}\right\}=\frac{at}{2}\sin at$$

$$\therefore\ L^{-1}\left[\frac{1}{a^2}\cdot\frac{s}{(s^2+a^2)^2}\cdot a^4\right]=\frac{at}{2}\sin at$$

$$\therefore\ L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\frac{1}{a}\cdot\frac{t}{2}\sin at$$

Ex. 2 : If $L^{-1}\left\{\dfrac{s^2+4}{(s^2-4)^2}\right\}=t\cos h2t$, find $L^{-1}\left\{\dfrac{s^2+9}{(s^2-9)^2}\right\}$.

Sol. : B the above rule, since $L^{-1}\left\{\dfrac{s^2+4}{(s^2-4)^2}\right\} = t\cos h\,2t$, replacing s by $\dfrac{s}{a}$ and t by at (after dividing the given function by a)

$$L^{-1}\left\{\frac{1}{a}\cdot\frac{(s/a)^2+4}{[(s/a)^2-4]^2}\right\} = at\cos h\,2at$$

$$\therefore\quad L^{-1}\left\{\frac{1}{a^3}\cdot\frac{s^2+4a^2}{(s^2-4a^2)^2}\cdot a^4\right\} = at\cos h\,2at$$

$$\therefore\quad L^{-1}\left\{\frac{s^2+4a^2}{(s^2-4a^2)^2}\right\} = t\cos h\,2at$$

Comparing the l.h.s. with the required result, we put $a = 3/2$.

$$\therefore\quad L^{-1}\left\{\frac{s^2+9}{(s^2-9)^2}\right\} = t\cos h\,3t.$$

EXERCISE

1. If $L^{-1}\left\{\dfrac{s}{(s^2+4)^2}\right\} = \dfrac{t}{4}\sin 2t$, find $L^{-1}\left\{\dfrac{s}{(s^2+1)^2}\right\}$. [**Ans. :** $\dfrac{t}{2}\sin t$]

2. If $L^{-1}\left\{\dfrac{s^2+1}{(s^2-1)^2}\right\} = t\cos h\,t$, find $L^{-1}\left\{\dfrac{s^2+4}{(s^2-4)^2}\right\}$. [**Ans. :** $t\cos h\,2t$]

3. If $L^{-1}\left\{\dfrac{1}{(s^2+4)^2}\right\} = \dfrac{1}{16}(\sin 2t - 2t\cos 2t)$,

   find $L^{-1}\left\{\dfrac{1}{(s^2+4a^2)^2}\right\}$ and also $L^{-1}\left\{\dfrac{1}{(s^2+9)^2}\right\}$.

   [**Ans. :** $\dfrac{1}{16a^3}(\sin 2at - 2at\cos 2at)$, $\dfrac{1}{54}(\sin 3t - 3t\cos 3t)$]

(e) Inverse by convolution theorem :

Definition : If $f_1(t)$ and $f_2(t)$ are two functions then the following integral

$$\int_0^t f_1(u)\,f_2(t-u)\,du$$

is called the convolution (= twisting, coiling, winding together) of $f_1(t)$ and $f_2(t)$ and is denoted by $f_1(t) * f_2(t)$. Thus,

$$f_1(t) * f_2(t) = \int_0^t f_1(t)\,f_2(t-u)\,du.$$

Theorem : Let $L\,f_1(t) = \Phi_1(s)$ and $L\,f_2(t) = \Phi_2(s)$, then

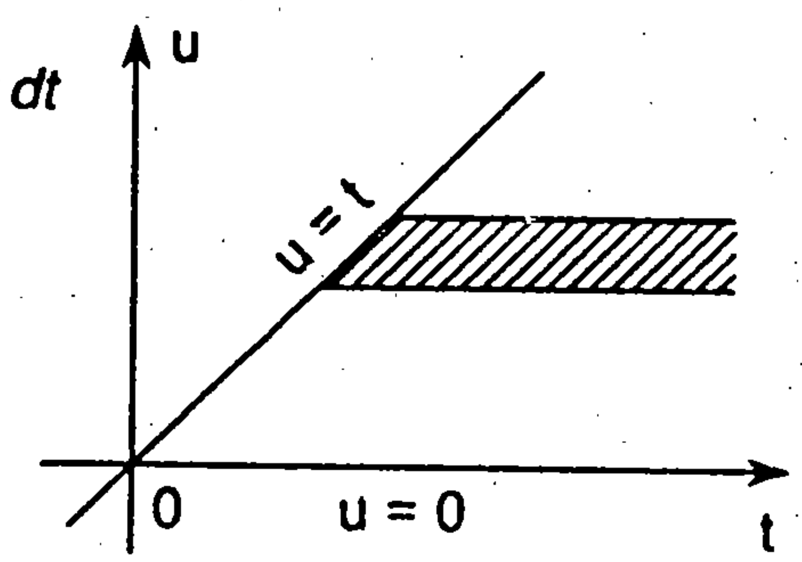
$$\boxed{L^{-1}[\Phi_1(s)\cdot\Phi_2(s)] = \int_0^t f_1(u)\cdot f_2(t-u)\,du} \qquad \text{.......... (16)}$$

where, $f_1(t) = L^{-1}\,\Phi_1(s)$ and $f_2(t) = L^{-1}\,\Phi_2(s)$

Proof. : Let $\Phi(t) = \int_0^t f_1(u)\cdot f_2(t-u)\,du.$

$$\therefore\quad L[\Phi(t)] = \int_0^\infty e^{-st}\,\Phi(t)\,dt = \int_0^\infty e^{-st}\int_0^t f_1(u)\cdot f_2(t-u)\,du\,dt$$

$$= \int_0^\infty\int_0^t e^{-st} f_1(u)\cdot f_2(t-u)\,du\,dt$$



On the r.h.s. integration is first carried out w.r.t. u and then w.r.t. Now, we change the order of integration. We integrate first with respect to t and then w.r.t. u. But now t varies from u to ∞ and then u varies from 0 to ∞.

$$\therefore\quad L[\Phi(t)] = \int_0^\infty\int_u^\infty e^{-st} f_1(u)\,f_2(t-u)\,dt\,du$$

$$= \int_0^\infty\int_u^\infty e^{-su}e^{-s(t-u)}\,f_1(u)\,f_2(t-u)\,dt\,du$$

$$= \int_0^\infty e^{-su}\,f_1(u)\,du\int_u^\infty e^{-s(t-u)}\,f_2(t-u)\,dt$$

Putting $t-u=p,\ dt=dp$

$$\therefore\quad L[\Phi(t)] = \int_0^\infty e^{-su}\,f_1(u)\,du\int_0^\infty e^{-sp}\,f_2(p)\,dp$$

$$\therefore\quad L\,\Phi(t) = L\,f_1(t)\cdot L\,f_2(t)$$

$$\therefore\quad L\Phi(t) = \Phi_1(s)\cdot\Phi_2(s)$$

$$\therefore\quad L^{-1}[\Phi_1(s)\cdot\Phi_2(s)] = \Phi(t)$$

$$= \int_0^t f_1(u)\cdot f_2(t-u)\,du$$

Cor. : If $\Phi_1(s) = \Phi(s)$ and $\Phi_2(s) = \dfrac{1}{s}$ then $f_2(t) = L^{-1}\,\Phi_2(s) = 1.$

$$\therefore\quad L^{-1}\left[\frac{1}{s}\Phi(s)\right] = \int_0^t f_1(u)\,du = \int_0^t L^{-1}\,\Phi(s)\,ds.$$

Note : 1. Taking Laplace transforms of both sides of (16), we get

$$\Phi_1(s)\,\Phi_2(s) = L\left[\int_0^t f_1(u)\,f_2(t-u)\,du\right]$$

But $\Phi_1(s) = L[f_1(t)]$ and $\Phi_2(s) = L[f_2(t)]$

$$\therefore \; L\left[\int_0^t f_1(u)\cdot f_2(t-u)\,du\right] = L\left[f_1(t)\right]\cdot L\left[f_2(t)\right]$$

This means the Laplace transform of the convolution of two functions is equal to the product of the Laplace transforms of the two functions.

Note : 2. The above result given in the corollary is obtained independently in § (h) page 4-44.

Procedure of Applying Convolution Theorem

To find $L^{-1}\,\Phi_1(s)\cdot\Phi_2(s)$:-

1. Find $L^{-1}\,\Phi_1(s) = f_1(u)$, say, putting u in place of t.
2. Find $L^{-1}\,\Phi_2(s) = f_2(u)$, say, putting u in place of t.
3. Find $L^{-1}\,[\Phi_1(s)\cdot\Phi_2(s)] = \int_0^t f_1(u)\,f_2(t-u)\,du.$

Ex. 1 : Find the inverse of the following by using convolution theorem.

(i) $\dfrac{s^2}{(s^2+a^2)^2}$ **(M.U. 1995, 2000, 02)**

(ii) $\dfrac{s^2}{(s^2-a^2)^2}$ **(M.U. 2002)** (iii) $\dfrac{1}{(s-3)(s+3)^2}$

Sol. : (i) Let $\Phi_1(s) = \dfrac{s}{s^2+a^2}$, $\Phi_2(s) = \dfrac{s}{s^2+a^2}$.

$$L^{-1}\,\Phi_1(s) = L^{-1}\frac{s}{s^2+a^2} = \cos at, \quad L^{-1}\,\Phi_2(s) = \cos at$$

$$\therefore \; L^{-1}[\Phi(s)] = L^{-1}\left[\left(\frac{s}{s^2+a^2}\right)\left(\frac{s}{s^2+a^2}\right)\right] = \int_0^t \cos au\cdot\cos a(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos at + \cos a(2u-t)]\,du$$

$$= \frac{1}{2}\left[u\cos at + \frac{1}{2a}\sin a(2u-t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos at + \frac{1}{2a}\sin at + \frac{1}{2a}\sin at\right]$$

$$= \frac{1}{2a}[\sin at + at\cos at]$$

(For another method See Ex. 1(i), page 4-43)

(ii) Let $\Phi_1(s) = \dfrac{s}{s^2-a^2}$, $\Phi_2(s) = \dfrac{s}{s^2-a^2}$

$$\therefore \; L^{-1}\,\Phi_1(s) = L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cos h\,at,$$

$$L^{-1}\,\Phi_2(s) = \cos h\,at$$

$$\therefore \; L^{-1}\,\Phi(s) = \int_0^t \cos h au\cdot\cos h a(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos h\,at + \cos h\,a(2u-t)]\,du$$

$$= \frac{1}{2}\left[u\cos h\,at + \frac{1}{2a}\sin h\,a(2u-t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos h\,at + \frac{1}{2a}\sin h\,at + \frac{1}{2a}\sin h\,at\right]$$

$$= \frac{1}{2}[\sin h\,at + at\cos h\,at]$$

(iii) Let $\Phi_1(s) = \dfrac{1}{s-3}$, $\Phi_2(s) = \dfrac{1}{(s+3)^2}$

$$\therefore \; L^{-1}\,\Phi_1(s) = L^{-1}\frac{1}{(s-3)} = e^{3t},$$

$$L^{-1}\,\Phi_2(s) = L^{-1}\frac{1}{(s+3)^2} = e^{-3t}\cdot L^{-1}\frac{1}{s^2} = e^{-3t}\cdot t$$

$$\therefore \; L^{-1}\Phi(s) = \int_0^t e^{3u}\cdot e^{-3(t-u)}\cdot(t-u)\,du$$

$$= \int_0^t e^{6u-3t}(t-u)\,du$$

$$= \left[(t-u)\frac{e^{6u-3t}}{6} - \left(\frac{e^{6u-3t}}{36}\right)(-1)\right]_0^t$$

$$= \left[\frac{e^{3t}}{36} - t\frac{e^{-3t}}{6} - \frac{e^{-3t}}{36}\right]$$

$$= \frac{1}{36}\left[e^{3t} - e^{-3t} - 6t\,e^{-3t}\right]$$

Ex. 2 : Find the inverse of the following by using convolution theorem.

(i) $\dfrac{1}{s^2(s+1)^2}$ **(M.U. 1997, 99)** (ii) $\dfrac{s^2}{(s^2+1)(s^2+4)}$ **(M.U. 2004)**

(iii) $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ **(M.U. 1996, 2004)**

(iv) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ **(M.U. 2004)** **(v)** $\dfrac{1}{(s+3)(s^2+2s+2)}$ **(M.U. 2004)**

Sol. : (i) Let $\Phi_1(s) = \dfrac{1}{(s+1)^2}$, $\Phi_2(s) = \dfrac{1}{s^2}$

$$\therefore \ L^{-1}\Phi_1(s) = L^{-1}\frac{1}{(s+1)^2} = e^{-t}L^{-1}\frac{1}{s^2} = e^{-t}\,t,\ L^{-1}\Phi_2(s) = L^{-1}\frac{1}{s^2} = t$$

$$\therefore \ L^{-1}\Phi(s) = L^{-1}\left[\frac{1}{(s+1)^2}\cdot\frac{1}{s^2}\right] = \int_0^t e^{-u}u\cdot(t-u)\,du$$

$$= t\int_0^t e^{-u}\cdot u\,du - \int_0^t e^{-u}\cdot u^2\,du$$

$$= t\Big[u(-e^{-u}) - (e^{-u})(1)\Big]_0^t - \Big[u^2(-e^{-u}) - (e^{-u})(2u) + (-e^{-u})(2)\Big]_0^t$$

$$= t\Big[-t\,e^{-t} - e^{-t} + 1\Big] - \Big[-t^2e^{-t} - 2t\,e^{-t} - 2e^{-t} + 2\Big]$$

$$= t\,e^{-t} + 2e^{-t} + t - 2$$

(ii) Let $\Phi_1(s) = \dfrac{s}{s^2+4}$, $\Phi_2(s) = \dfrac{s}{s^2+1}$

$$L^{-1}\Phi_1(s) = L^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t,\quad L^{-1}\Phi_2(s) = L^{-1}\left(\frac{s}{s^2+1}\right) = \cos t$$

$$\therefore \ L^{-1}\Phi(s) = L^{-1}\left[\frac{s}{s^2+4}\cdot\frac{s}{s^2+1}\right] = \int_0^t \cos 2u\cdot\cos(t-u)\,du$$

$$= \frac{1}{2}\int_0^t[\cos(u+t) + \cos(3u-t)]\,du$$

$$= \frac{1}{2}\left[\sin(u+t) + \frac{1}{3}\sin(3u-t)\right]_0^t$$

$$= \frac{1}{2}\left[\sin 2t + \frac{1}{3}\sin 2t - \sin t + \frac{1}{3}\sin t\right]$$

$$= \frac{1}{2}\left[\frac{4}{3}\sin 2t - \frac{2}{3}\sin t\right] = \frac{1}{3}[2\sin 2t - \sin t] \qquad \text{.......... (A)}$$

[See that we have solved this example by another method on page 4-12 Ex. 2 (iii)]

(iii) $$L^{-1}\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = L^{-1}\frac{(s+1)^2+2}{[(s+1)^2+1][(s+1)^2+4]}$$

$$= e^{-t}L^{-1}\frac{s^2+2}{(s^2+1)(s^2+4)}$$

$$= e^{-t}\cdot L^{-1}\frac{s^2}{(s^2+1)(s^2+4)} + e^{-t}L^{-1}\left[\frac{2}{(s^2+1)(s^2+4)}\right]$$

$$= e^{-t}\cdot L^{-1}\frac{s^2}{(s^2+1)(s^2+4)} + \frac{2e^{-t}}{3}\cdot L^{-1}\left[\frac{1}{(s^2+1)} - \frac{1}{(s^2+4)}\right]$$

(By the result obtained above in (A))

$$= \frac{e^{-t}}{3}[2\sin 2t - \sin t] + \frac{2}{3}e^{-t}\left[\sin t - \frac{1}{2}\sin 2t\right]$$

$$= \frac{e^{-t}}{3}[\sin 2t + \sin t]$$

[See that we have solved this example by another method on page 4-15 Ex. 2 (viii)]

(iv) $$L^{-1}\left\{\frac{(s+2)^2}{(s^2+4s+8)^2}\right\} = L^{-1}\left\{\frac{(s+2)^2}{(s+2)^2+2^2}\right\} = e^{-2t}L^{-1}\frac{s^2}{(s^2+2^2)^2}$$

$$L^{-1}\frac{s}{s^2+2^2} = \cos 2t$$

$$\therefore \ L^{-1}\left[\left(\frac{s}{s^2+2^2}\right)\left(\frac{s}{s^2+2^2}\right)\right] = \int_0^t \cos 2u\cdot\cos 2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t[\cos 2t + \cos(4u-2t)]\,du$$

$$= \frac{1}{2}\left[u\cos 2t + \frac{1}{4}\sin(4u-2t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{4}\sin 2t\right]$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{2}\sin 2t\right]$$

$$\therefore \ L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right] = \frac{e^{-2t}}{2}\left[t\cos 2t + \frac{1}{2}\sin 2t\right]$$

$$= \frac{e^{-2t}}{4}[2t\cos 2t + \sin 2t]$$

(v) $$L^{-1}\left[\frac{1}{(s+3)(s^2+2s+2)}\right] = L^{-1}\left[\frac{1}{[(s+1)+2][(s+1)^2+1]}\right]$$

$$= e^{-t}L^{-1}\left[\frac{1}{(s+2)(s^2+1)}\right]$$

Let $\Phi_1(s) = \dfrac{1}{s^2+1}$ and $\Phi_2(s) = \dfrac{1}{s+2}$

$\therefore \; L^{-1}[\Phi_1(s)] = \sin at$ and $L^{-1}[\Phi_2(s)] = e^{-2t}$

$$\therefore \; L^{-1}[\Phi_1(s)\cdot\Phi_2(s)] = \int_0^t \sin u \cdot e^{-2(t-u)}du$$

$$= \int_0^t \sin u \cdot e^{-2t} \cdot e^{2u}\, du$$

$$= e^{-2t}\int_0^t e^{2u} \cdot \sin u\, du$$

$$= e^{-2t} \cdot \frac{1}{4+1}\left[e^{2u}(2\sin u - \cos u)\right]_0^t$$

$$= \frac{1}{5}e^{-2t}\left[e^{2t}(2\sin t - \cos t) + 1\right]$$

$$\therefore \; L^{-1}\left[\frac{1}{(s+3)(s^2+2s+2)}\right] = e^{-t}\cdot L^{-1}[\Phi_1(s)\,\Phi_2(s)]$$

$$= e^{-t}\cdot\frac{1}{5}\cdot e^{-2t}\left[e^{2t}(2\sin t - \cos t) + 1\right]$$

$$= \frac{1}{5}\cdot e^{-3t}\left[e^{2t}(2\sin t - \cos t) + 1\right]$$

$$= \frac{1}{5}\left[e^{-t}(2\sin t - \cos t) + e^{-3t}\right]$$

Ex. 3 : Using convolution theorem find the inverse Laplace transform of the following.

(i) $\dfrac{1}{(s-2)^4(s+3)}$ **(M.U. 1993, 98, 99)**

(ii) $\dfrac{1}{(s^2+4s+13)^2}$ **(M.U. 2002)**

(iii) $\dfrac{(s+3)^2}{(s^2+6s+5)^2}$

(iv) $\dfrac{1}{(s-2)(s+2)^2}$ **(M.U. 1994)**

(v) $\dfrac{s}{(s^2+4)(s^2+1)}$ **(M.U. 1995, 2002, 03)**

Sol. : (i) Let $\Phi_1(s) = \dfrac{1}{s+3}$, $\Phi_2(s) = \dfrac{1}{(s-2)^4}$

$$\therefore \; L^{-1}\Phi_1(s) = e^{-3t}, \quad L^{-1}\Phi_2(s) = e^{2t}\,L^{-1}\frac{1}{s^4} = e^{2t}\cdot\frac{t^3}{6}$$

$$\therefore \; L^{-1}\Phi(s) = \int_0^t e^{-3u}\cdot e^{2(t-u)}\cdot\frac{(t-u)^3}{6}du$$

$$= \int_0^t e^{(2t-5u)}\cdot\frac{(t-u)^3}{6}du = e^{2t}\int_0^t e^{-5u}\,\frac{(t-u)^3}{6}du$$

$$= e^{2t}\left[\frac{(t-u)^3}{6}\left(\frac{-e^{-5u}}{5}\right) - \frac{(t-u)^2}{2}(-1)\left(\frac{e^{-5u}}{25}\right)\right.$$
$$\left. + (t-u)(+1)\left(\frac{-e^{-5u}}{125}\right) - (-1)\left(\frac{e^{-5u}}{625}\right)\right]_0^t$$

$$= e^{2t}\left[\frac{e^{-5t}}{625} - \left\{-\frac{t^3}{30} + \frac{t^2}{50} - \frac{t}{125} + \frac{1}{625}\right\}\right]$$

$$= \frac{e^{-3t}}{625} - e^{2t}\left[\frac{1}{625} - \frac{t}{125} + \frac{t^2}{50} - \frac{t^3}{30}\right]$$

(ii) $$L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]^2$$

$$= e^{-2t}\cdot L^{-1}\left[\frac{1}{(s^2+3^2)}\right]^2$$

We find $L^{-1}\left[\dfrac{1}{(s^2+3^2)^2}\right]$ by convolution theorem.

Let $\Phi_1(s) = \dfrac{1}{s^2+3^2}$ and $\Phi_2(s) = \dfrac{1}{s^2+3^2}$ and $\Phi(s) = \dfrac{1}{(s^2+3^2)^2}$

$\therefore \; L^{-1}[\Phi_1(s)] = \dfrac{1}{3}\sin 3t$ and $L^{-1}[\Phi_2(s)] = \dfrac{1}{3}\sin 3t$

$$\therefore \; L^{-1}[\Phi(s)] = \frac{1}{9}\int_0^t \sin 3u\cdot\sin 3(t-u)\,du$$

$$= -\frac{1}{18}\int_0^t[\cos 3t - \cos(6u-3t)]\,du$$

$$= -\frac{1}{18}\left[u\cos 3t - \frac{\sin(6u-3t)}{6}\right]_0^t$$

$$= -\frac{1}{18}\left[t\cos 3t - \frac{\sin 3t}{6} - \frac{\sin 3t}{6}\right]$$

$$= \frac{1}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

$$\therefore \; L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = \frac{e^{-2t}}{18}\left[\frac{\sin 3t}{3} - t\cos t3t\right]$$

(iii) $L^{-1}\dfrac{(s+3)^2}{(s^2+6s+5)^2} = L^{-1}\dfrac{(s+3)^2}{\left[(s+3)^2-2^2\right]^2}$

$= e^{-3t} \cdot L^{-1}\dfrac{s^2}{(s^2-2^2)^2}$ (By first shifting theorem)

We find $L^{-1}\dfrac{s^2}{(s^2-2^2)^2}$ by convolution theorem.

Let $\Phi_1(s) = \dfrac{s}{s^2-2^2}$, $\Phi_2(s) = \dfrac{s}{s^2-2^2}$.

$\therefore \; L^{-1}\Phi_1(s) = \cos h 2u, \quad L^{-1}\Phi_2(s) = \cos h 2u$

$$\therefore \; L^{-1}\Phi(s) = \int_0^t \cos h 2u \cdot \cos h 2(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\cos h 2t + \cos h 2(2u-t)]\, du$$

$$= \frac{1}{2}\left[u \cos h 2t + \frac{\sin h 2(2u-t)}{4}\right]_0^t$$

$$= \frac{1}{2}\left[t \cos h 2t + \frac{1}{4}\sin h 2t + \frac{1}{4}\sin h 2t\right]$$

$$= \frac{1}{4}[2t \cos h 2t + \sin h 2t]$$

$$\therefore \; L^{-1}\frac{(s+3)^2}{(s^2+6s+5)^2} = \frac{1}{4}e^{-3t}[2t\cosh 2t + \sinh 2t]$$

(iv) Let $\Phi_1(s) = \dfrac{1}{s-2}$, $\Phi_2(s) = \dfrac{1}{(s+2)^2}$

$\therefore \; L^{-1}\Phi_1(s) = e^{2t}, \quad L^{-1}\Phi_2(s) = e^{-2t} \cdot L^{-1}\dfrac{1}{s^2} = e^{-2t} \cdot t$

$$\therefore \; L^{-1}\Phi(s) = \int_0^t e^{2u} \cdot e^{-2(t-u)} \cdot (t-u)\, du$$

$$= \int_0^t e^{4u-2t}(t-u)\, du$$

$$= \left[(t-u) \cdot \frac{e^{4u-2t}}{4}\right]_0^t - \int_0^t \frac{e^{4u-2t}}{4} \cdot (-1)\, du$$

$$= \left[(t-u) \cdot \frac{e^{4u-2t}}{4} + \frac{e^{4u-2t}}{16}\right]_0^t$$

$$= \frac{e^{2t}}{16} - \frac{t\,e^{-2t}}{4} - \frac{e^{-2t}}{16}$$

$$= \frac{1}{16}\left[e^{2t} - e^{-2t} - 4\,t\,e^{-2t}\right]$$

(v) Let $\Phi_1(s) = \dfrac{s}{s^2+4}$, $\Phi_2(s) = \dfrac{1}{s^2+1}$

$\therefore \; L^{-1}\Phi_1(s) = \cos 2t, \quad L^{-1}\Phi_2(s) = \sin t$

$$\therefore \; L^{-1}\Phi(s) = \int_0^t \cos 2u \cdot \sin(t-u)\, du$$

$$= \frac{1}{2}\int_0^t [\sin(u+t) + \sin(t-3u)]\, du$$

$$= \frac{1}{2}\left[-\cos(u+t) + \frac{1}{3}\cos(t-3u)\right]_0^t$$

$$= \frac{1}{2}\left[-\cos 2t + \frac{1}{3}\cos 2t + \cos t - \frac{1}{3}\cos t\right]$$

$$= \frac{1}{2}\left[\frac{2}{3}\cos t - \frac{2}{3}\cos 2t\right] = \frac{1}{3}[\cos t - \cos 2t]$$

[See that we have solved the example by another method on page 4-13 Ex. 2 (iv)]

Ex. 4 : Find the inverse Laplace transform of the following by convolution theorem.

(i) $\dfrac{1}{(s+3)(s-1)}$ **(ii)** $\dfrac{s}{(s^2+a^2)^2}$ **(M.U. 1984, 87, 2003)**

(iii) $\dfrac{1}{s(s^2-a^2)}$ **(iv)** $\dfrac{1}{s\sqrt{s+4}}$ **(M.U. 2002)** **(v)** $\dfrac{1}{s}\log\left(\dfrac{s+3}{s+2}\right)$

(vi) $\dfrac{1}{s}\log\left(1+\dfrac{1}{s^2}\right)$ **(M.U. 1996)** **(vii)** $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$ **(M.U. 2002)** **(viii)** $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$

Sol. : (i) $L^{-1}\left(\dfrac{1}{s+3}\right) = e^{-3u}$, $L^{-1}\left(\dfrac{1}{s-1}\right) = e^{u}$

$$\therefore \; L^{-1}\left[\frac{1}{(s+3)} \cdot \frac{1}{s+1}\right] = \int_0^t e^{-3u} \cdot e^{t-u}\, du$$

$$= e^t \int_0^t e^{-4u}\, du = e^t\left[\frac{e^{-4u}}{-4}\right]_0^t = \frac{e^t}{4}\left[1 - e^{-4t}\right]$$

(ii) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos au,\quad L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin au$

$$\therefore\quad L^{-1}\left[\frac{s}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)}\right] = \int_0^t \cos au\,\frac{1}{a}\sin a(t-u)\,du$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin a(t-2u)]\,du$$

$$= \frac{1}{2a}\left[\sin at\cdot u + \frac{1}{2a}\cos a(t-2u)\right]_0^t$$

$$= \frac{1}{2a}\cdot t\cdot \sin at.$$

(iii) $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sin h\,au,\quad L^{-1}\cdot\frac{1}{s} = 1$

$$\therefore\quad L^{-1}\left[\frac{1}{(s^2-a^2)}\cdot\frac{1}{s}\right] = \int_0^t \frac{1}{a}\sin h\,au\cdot 1\cdot du$$

$$= \frac{1}{a^2}\left[\cos h\,au\right]_0^t = \frac{1}{a^2}[\cos h\,at - 1]$$

(iv) $L^{-1}\frac{1}{(s+4)^{1/2}} = e^{-4u}\cdot L^{-1}\frac{1}{s^{1/2}} = e^{-4u}\cdot\frac{u^{-1/2}}{\overline{|1/2}}$

$L^{-1}\frac{1}{s} = 1$

$$\therefore\quad L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2-a^2}\right] = \int_0^t \frac{e^{-4u}\cdot u^{-1/2}}{\sqrt{\pi}}\cdot 1\cdot du$$

Put $4u = x^2,\ du = \frac{x\,dx}{2},\ \sqrt{u} = x/2$

$$\therefore\quad L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2-a^2}\right] = \int_0^{2\sqrt{t}} \frac{e^{-x^2}}{\sqrt{\pi}}\,dx = \frac{1}{2}\,erf(2\sqrt{t})$$

(v) $L^{-1}\log\left(\frac{s+3}{s+2}\right) = -\frac{1}{u}L^{-1}\left\{\frac{d}{ds}\log\left(\frac{s+3}{s+2}\right)\right\}$ [By (17), page 4-31]

$$= -\frac{1}{u}L^{-1}\left[\frac{d}{ds}\{\log(s+3) - \log(s+2\}\right]$$

$$= -\frac{1}{u}L^{-1}\left[\frac{1}{u+3} - \frac{1}{u+2}\right] = -\frac{1}{u}\left(e^{-3u} - e^{-2u}\right) \quad \text{And } L^{-1}\frac{1}{s} = 1$$

$$\therefore\quad L^{-1}\frac{1}{s}\log\left(\frac{s+3}{s+2}\right) = \int_0^t \frac{e^{-2u} - e^{-3u}}{u}\cdot du.$$ [By (16), page 4-19]

(vi) $L^{-1}\log\left(1+\frac{1}{s^2}\right) = -\frac{1}{u}L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2+1}{s^2}\right)\right]$ [By (17), page 4-31]

$$= -\frac{1}{u}L^{-1}\left[\frac{d}{ds}\{\log(s^2+1) - \log s^2\}\right]$$

$$= -\frac{1}{u}L^{-1}\left[\frac{2s}{s^2+1} - \frac{2}{s}\right] = -\frac{2}{u}(\cos u - 1) \quad \text{And } L^{-1}\frac{1}{s} = 1$$

$$\therefore\quad L^{-1}\left[\frac{1}{s}\cdot\log\left(1+\frac{1}{s^2}\right)\right] = \int_0^t -\frac{2}{u}(\cos u - 1)\cdot 1\cdot du$$

[By (16), page 4-19]

(vii) Let $\Phi_1(s) = \frac{s+1}{s^2+2s+2}$ and $\Phi_2(s) = \frac{s}{s^2+1}$

$$L^{-1}\frac{s+1}{(s+1)^2+1} = e^{-u}L^{-1}\frac{s}{s^2+1} = e^{-u}\cos u,\quad L^{-1}\frac{s}{s^2+1} = \cos u$$

$$\therefore\quad L^{-1}\left[\frac{s+1}{(s+1)^2+1}\cdot\frac{s}{s^2+1}\right] = \int_0^t e^{-u}\cos u\cdot\cos(u-t)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}[\cos(2u-t) + \cos t]\,du$$

$$= \frac{1}{2}\left[\frac{1}{5}e^{-u}[-\cos(2u-t) + 2\sin(2u-t)] - e^{-u}\cos t\right]_0^t$$

[By (A) page 3-2]

$$= \frac{1}{2}\left[\frac{1}{5}e^{-t}\{-\cos t + 2\sin t\} - e^{-t}\cos t - \frac{1}{5}\{-\cos t - 2\sin t\} + \cos t\right]$$

$$= \frac{1}{10}\left[e^{-t}(2\sin t - 6\cos t) + (2\sin t + 6\cos t)\right]$$

(viii) $L^{-1}\frac{(s+2)^2}{(s^2+4s+8)^2} = L^{-1}\frac{(s+2)^2}{[(s+2)^2+2^2]^2} = e^{-2t}L^{-1}\frac{s^2}{(s^2+2^2)^2}$

Let $\Phi_1(s) = \frac{s}{s^2+2^2},\quad \Phi_2(s) = \frac{s}{s^2+2^2}$

$$\therefore\quad L^{-1}\Phi_1(s) = L^{-1}\frac{s}{s^2+2^2} = \cos 2t,\quad L^{-1}\Phi_2(s) = \cos 2t$$

$$\therefore\quad L^{-1}\Phi(s) = L^{-1}\left[\left(\frac{s}{s^2+2^2}\right)\left(\frac{s}{s^2+2^2}\right)\right] = \int_0^t \cos 2u\cdot\cos 2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos 2t + \cos 2(2u-t)]\,du$$

$$= \frac{1}{2}\left[u\cos 2t + \frac{1}{4}\sin 2(2u-t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{4}\sin 2t\right]$$

$$= \frac{1}{4}[\sin 2t + 2t\cos 2t]$$

$$\therefore \quad L^{-1}\frac{(s+2)^2}{(s^2+4s+8)^2} = \frac{1}{4}e^{-2t}[\sin 2t + 2t\cos 2t]$$

EXERCISE

Using convolution theorem find the inverse Laplace transform of,

1. $\dfrac{s^2}{(s^2+2^2)^2}$ **(M.U. 1988)** 2. $\dfrac{s^2}{(s^2+9)^2}$ 3. $\dfrac{s^2}{(s^2+9)(s^2+4)}$

4. $\dfrac{s}{(s^2+1)^2}$ 5. $\dfrac{s}{(s^2+4)^2}$ 6. $\dfrac{1}{s(s^2+4)}$ 7. $\dfrac{a}{s(s-a)}$

8. $\dfrac{(s+3)^2}{(s^2+6s+18)^2}$ 9. $\dfrac{(s+5)^2}{(s^2+10s+16)^2}$ 10. $\dfrac{1}{(s-3)(s+4)^2}$

11. $\dfrac{s^2}{(s^2+2^2)^2}$ **(M.U. 1996)** 12. $\dfrac{s+3}{(s^2+6s+13)^2}$ **(M.U. 2004)**

13. $\dfrac{s}{s^4+8s^2+16}$ **(M.U. 2003)** 14. $\dfrac{s}{(s^2-a^2)^2}$ **(M.U. 2003)**

15. $\dfrac{2s}{(s^2+1)^2}$ **(M.U. 2004)**

[**Ans. : (1)** $\frac{1}{4}(\sin 2t + 2t\cos 2t)$, **(2)** $\frac{1}{6}(\sin 3t + 3t\cos 3t)$,

(3) $\frac{1}{5}(3\sin 3t - 2\sin 2t)$, **(4)** $\frac{1}{2}t\sin t$, **(5)** $\frac{1}{4}t\sin 2t$,

(6) $\frac{1}{4}(1-\cos 2t)$, **(7)** $e^{at}-1$, **(8)** $\frac{1}{6}e^{-3t}\cdot[\cos 3t + 3t\sin 3t]$,

(9) $\frac{1}{6}e^{-5t}\cdot[\sin h\,3t + 3t\cos h\,3t]$, **(10)** $\frac{1}{49}\left[e^{3t} - e^{-4t} - 7t\,e^{-4t}\right]$,

(11) $\frac{1}{4}(\sin 2t + 2t\cos 2t)$ **(12)** $\frac{1}{4}e^{-3t}\cdot t\sin 2t$, **(13)** $\frac{1}{4}t\sin 2t$,

(14) $\frac{1}{2a}[at\cos h\,at + \sin h\,at]$, **(15)** $t\sin t$.]

(f) Use of differentiation of $\Phi(s)$: We know that (§ 8, page 3-25) if $L[f(t)] = \Phi(s)$, then $-\Phi'(s) = L[t\,f(t)]$

i.e. if $L\left[\frac{-1}{t}f(t)\right] = \Phi(s)$, then $\Phi'(s) = L[f(t)]$.

Hence, if $L^{-1}[\Phi'(s)] = f(t)$, then $L^{-1}[\Phi(s)] = -\frac{1}{t}f(t)$.

i.e. $$\boxed{L^{-1}\Phi(s) = -\frac{1}{t}L^{-1}[\Phi'(s)] \quad i.e. \quad L^{-1}[\Phi'(s)] = -t\,L^{-1}\Phi(s)} \quad \text{.......... (17)}$$

Remark 1 : The result can be extended further.

$$L^{-1}\Phi(s) = -\frac{1}{t^2}L^{-1}[\Phi''(s)] \quad \text{or} \quad L^{-1}[\Phi''(s)] = t^2\cdot L^{-1}[\Phi(s)]$$

Remark 2 : These results can be profitably used to find $L^{-1}\Phi(s)$ if we know $L^{-1}\Phi'(s)$ *i.e.* if $\Phi'(s)$ comes out to be a standard result. (See ex. 1) or to find $L^{-1}\Phi'(s)$ if we know $L^{-1}\Phi(s)$ *i.e.* the given function is the derivative of a standard result. (See Ex. 1, page 4-36).

Ex. 1 : Find inverse Laplace transform of,

(i) $\log\left(\frac{s+a}{s+b}\right)$ **(M.U. 1993, 96)** **(ii)** $\log\left(1+\frac{a^2}{s^2}\right)$ **(M.U. 1999, 2002)** **(iii)** $\log\left[\frac{s^2+a^2}{\sqrt{s+b}}\right]$ **(M.U. 2003, 04)**

(iv) $\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$ **(v)** $2\tan h^{-1}s$ **(M.U. 2003)** **(vi)** $s\log\left(\frac{s+1}{s-1}\right)$

Sol. : We have $L^{-1}[\Phi(s)] = -\frac{1}{t}L^{-1}[\Phi'(s)]$

(i) $\therefore$ $$L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\{\log(s+a) - \log(s+b)\}\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right] = -\frac{1}{t}(e^{-at} - e^{-bt})$$

(ii) $$L^{-1}\left[\log\left(1+\frac{a^2}{s^2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(1+\frac{a^2}{s^2}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{a^2+s^2}{s^2}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left\{\log(a^2+s^2) - \log s^2\right\}\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{a^2+s^2} - \frac{2}{s}\right]$$

$$= -\frac{1}{t}[2\cos at - 2\cdot 1] = \frac{2}{t}[1-\cos at]$$

(iii) $$L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left\{\log(s^2+a^2) - \frac{1}{2}\log(s+b)\right\}\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{1}{2}\cdot\frac{1}{s+b}\right]$$

$$= -\frac{1}{t}\left[2\cos at - \frac{1}{2}e^{-bt}\right] = \frac{1}{t}\left[\frac{1}{2}e^{bt} - 2\cos at\right]$$

(iv) $$L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}\right]$$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\left\{\log(s^2+a^2) - \log(s^2+b^2)\right\}\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right]$$

$$= -\frac{1}{t}[2\cos at - 2\cos bt] = \frac{2}{t}(\cos bt - \cos at)$$

(v) $$L^{-1}(2\tan h^{-1}s) = L^{-1}\left[2\cdot\frac{1}{2}\log\left(\frac{1+s}{1-s}\right)\right]$$

$$= L^{-1}\left[\log\left(\frac{1+s}{1-s}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{1+s}{1-s}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{1+s} + \frac{1}{1-s}\right] = -\frac{1}{t}L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= -\frac{1}{t}\left[e^{-t} - e^{t}\right] = \frac{2}{t}\left(\frac{e^t - e^{-t}}{2}\right) = \frac{2}{t}\sin ht$$

(vi) $$L^{-1}\left[s\log\left(\frac{s+1}{s-1}\right)\right] = -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}s\log\left(\frac{s+1}{s-1}\right)\right]$$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\{s\log(s+1) - s\log(s-1)\}\right]$$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{s}{s+1} + \log(s+1) - \frac{s}{s-1} - \log(s-1)\right]$$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{s}{s+1} - \frac{s}{s-1}\right] - \frac{1}{t}\cdot L^{-1}[\log(s+1) - \log(s-1)]$$

$$= -\frac{1}{t}\cdot L^{-1}\left(-\frac{2s}{s^2-1}\right) - \frac{1}{t}\cdot\left(-\frac{1}{t}\right)\cdot L^{-1}\left\{\frac{d}{ds}[\log(s+1) - \log(s-1)]\right\}$$

$$= \frac{2}{t}\cdot L^{-1}\left(\frac{s}{s^2-1}\right) + \frac{1}{t^2}\cdot L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= \frac{2}{t}\cos ht + \frac{1}{t^2}(e^{-t} - e^{t})$$

$$= \frac{2}{t}\cos ht - \frac{2}{t^2}\left(\frac{e^t - e^{-t}}{2}\right) = \frac{2}{t}\cos ht - \frac{2}{t^2}\sin ht$$

Note: Note $L^{-1}\left(\frac{s}{s+1}\right)$ and $L^{-1}\left(\frac{s}{s-1}\right)$ do not exist. Hence, we have combined these two terms as above.

Ex. 2 : Find inverse Laplace transform of

(i) $\tan^{-1}\left(\frac{2}{s^2}\right)$ **(M.U. 1995, 2004)** **(ii)** $\cot^{-1} as$

(v) tan $\left(\frac{\quad}{b}\right)$ (M.U. 2004, 05)

Sol. : We have $L^{-1}[\Phi(s)] = -\frac{1}{t}L^{-1}[\Phi'(s)]$

(i) $\therefore$ $$L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left(\tan^{-1}\frac{2}{s^2}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{1+(4/s^4)}\left(-\frac{4}{s^3}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[-\frac{4s}{s^4+4}\right] = \frac{4}{t}L^{-1}\left[\frac{s}{s^2+4}\right]$$

$$= \frac{4}{t}\cdot L^{-1}\left[\frac{s}{(s^2+2)^2-(2s)^2}\right]$$

$$= \frac{4}{t}\cdot\frac{1}{4}L^{-1}\left[\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right]$$

$$= \frac{1}{t}\cdot L^{-1}\left[\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{t}\cdot\left[e^t\cdot L^{-1}\left(\frac{1}{s^2+1}\right)-e^{-t}\cdot L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$= \frac{1}{t}\left[e^t\sin t - e^{-t}\sin t\right]$$

$$= \frac{2\sin t}{t}\left(\frac{e^t-e^{-t}}{2}\right) = 2\sin t\sin ht$$

(ii) $L^{-1}[\cot^{-1}(as)] = -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\cot^{-1}(as)\right]$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{-a}{1+a^2s^2}\right] = \frac{a}{t}\cdot L^{-1}\left(\frac{1}{1+a^2s^2}\right)$$

$$= \frac{a}{t}\cdot L^{-1}\left[\frac{1}{a^2}\cdot\frac{1}{s^2+(1/a)^2}\right]$$

$$= \frac{1}{t}\cdot L^{-1}\left[\frac{1/a}{s^2+(1/a)^2}\right]$$

$$= \frac{1}{t}\sin\frac{t}{a}.$$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{1}{1+(4/s)^2}\left(-\frac{4}{s^2}\right)\right] = \frac{2}{t}\cdot L^{-1}\left(\frac{2}{s^2+4}\right)$$

$$= \frac{2}{t}\cdot\frac{1}{2}\sin 2t = \frac{1}{t}\sin 2t$$

(iv) $L^{-1}\left[\cot^{-1}(s+1)\right] = -\frac{1}{t}\frac{d}{ds}\left[\cot^{-1}(s+1)\right]$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{-1}{1+(s+1)^2}\right] = \frac{1}{t}\cdot L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{t}\cdot e^{-t}\cdot L^{-1}\left(\frac{1}{s^2+1}\right) = \frac{1}{t}e^{-t}\sin t.$$

(v) $\Phi(s) = \tan^{-1}\left(\frac{s+a}{b}\right)$

$\therefore\quad \Phi'(s) = \frac{1}{b}\cdot\frac{1}{1+[(s+a)/b]^2} = \frac{1}{b}\cdot\frac{b^2}{[b^2+(s+a)^2]} = \frac{b}{(s+a)^2+b^2}$

$\therefore\quad L^{-1}[\Phi'(s)] = e^{-at}\sin bt$

$\therefore\quad L^{-1}[\Phi(s)] = -\frac{1}{t}L^{-1}[\Phi'(s)] = -\frac{1}{t}\cdot e^{-at}\sin bt.$

EXERCISE

Find the inverse Laplace transform of,

1. $\log\left[\frac{s^2-4}{(s-3)^2}\right]$ 2. $\log\left[1+\frac{4}{s^2}\right]$ 3. $\frac{1}{2}\log\left[\frac{s-1}{s+1}\right]$ 4. $\log\left[1+\frac{1}{s^2}\right]$

(M.U. 2003)

5. $\frac{1}{2}\log\left(\frac{s^2+1}{s^2+4}\right)$ 6. $\frac{1}{2}\log\left(1-\frac{a^2}{s^2}\right)$ 7. $\tan^{-1}(s+1)$

(M.U. 1999)

8. $\cot^{-1}\left(\frac{2}{s^2}\right)$ 9. $\log\sqrt{\frac{s^2+1}{s^2}}$ 10. $\log\left[\frac{s^2+a^2}{(s+b)^2}\right]$

(M.U. 1998) **(M.U. 2003, 04)**

11. $\log\left(1+\frac{a}{s}\right)$ **(M.U. 2003)** 12. $\tan^{-1}\frac{1}{s}$ **(M.U. 2003)** 13. $\cot^{-1}s$ **(M.U. 1996, 98)**

14. $\log\sqrt{\frac{s^2+a^2}{s^2}}$ **(M.U. 1995, 98, 2004)** 15. $\log\left(1-\frac{1}{s^2}\right)$ **(M.U. 1997)**

16. $\tan^{-1}\left(\frac{s}{2}\right)$ **(M.U. 2004)**

[**Ans.:** (1) $\frac{2}{t}\left[e^{3t}-\cos h2t\right]$, (2) $\frac{2}{t}[1-\cos 2t]$, (3) $-\frac{1}{t}\sin ht$,

(4) $\frac{2}{t}(1-\cos t)$, (5) $\frac{1}{t}(\cos 2t-\cos t)$, (6) $\frac{1}{t}(1-\cos hat)$,

(7) $-\frac{1}{t}e^{-t}\sin t$, (8) $-\frac{2}{t}[\sin t\sin ht]$, (9) $\frac{1}{t}(1-\cos t)$

(10) $\frac{2}{t}(e^{-bt}-\cos at)$, (11) $\frac{1}{t}(1-e^{-at})$, (12) $\frac{1}{2t}\sin 2t$,

(13) $\frac{1}{t}\sin t$, (14) $\frac{1}{t}(1-\cos at)$, (15) $\frac{2}{t}(1-\cos ht)$,

(16) $-\frac{1}{t}\sin 2t$.]

Ex. 1 : Find the inverse Laplace transform of,

(i) $\frac{s}{(s^2+a^2)^2}$ **(M.U. 2000)** (ii) $\frac{s+3}{(s^2+6s+13)^2}$ **(M.U. 2004)**

Sol. : (i) Let $\Phi(s) = \frac{1}{s^2+a^2}$ $\quad\therefore\quad \Phi'(s) = \frac{-2s}{(s^2+a^2)^2}$

But $L^{-1}\Phi(s) = \frac{1}{a}\sin at$ and $L^{-1}\Phi'(s) = -t\,L^{-1}\Phi(s)$

[See (17), page 4-31]

$$\therefore\quad L^{-1}\left[-\frac{2s}{(s^2+a^2)^2}\right] = -t\cdot\frac{1}{a}\sin at$$

$$\therefore\quad L^{-1}\frac{s}{(s^2+a^2)^2} = \frac{1}{2a}\cdot t\cdot\sin at$$

Alternatively we can use the result 15 obtained on page 3-36. We have

$$L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \Phi(s)\,ds$$

$$\therefore\quad \frac{1}{t}f(t) = L^{-1}\int_s^\infty \Phi(s)\,ds$$

$$\therefore\quad f(t) = t\cdot L^{-1}\int_s^\infty \Phi(s)\,ds$$

But $f(t) = L^{-1}[\Phi(s)]$

$$\therefore\quad \boxed{L^{-1}[\Phi(s)] = t\cdot L^{-1}\left[\int_s^\infty \Phi(s)\,ds\right]} \qquad \text{.......... (A)}$$

$$\therefore\quad L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = t\cdot L^{-1}\left[\int_s^\infty \frac{s}{(s^2+a^2)^2}\,ds\right]$$

To find the integral, put $s^2+a^2 = z$

$$\therefore\quad \int\frac{s}{(s^2+a^2)^2}\,ds = \frac{1}{2}\int\frac{dz}{z^2} = -\frac{1}{2z} = -\frac{1}{2(s^2+a^2)}$$

$$\therefore\quad L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2}t\cdot L^{-1}\left(\frac{1}{s^2+a^2}\right)$$

$$= \frac{t}{2}\cdot\frac{1}{a}\sin at$$

(ii) $L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right] = L^{-1}\frac{(s+3)}{[(s^2+6s+9)+4]^2} = L^{-1}\frac{(s+3)}{[(s+3)^2+4]^2}$

$$= e^{-3t}\cdot L^{-1}\frac{s}{(s^2+4)^2} \qquad \text{(By shifting theorem)}$$

Let $\Phi(s) = \frac{1}{s^2+4}$ $\quad\therefore\quad \Phi'(s) = \frac{-2s}{(s^2+4)^2}$

But $L^{-1}[\Phi(s)] = \frac{1}{2}\sin 2t$ and $L^{-1}[\Phi'(s)] = -t\,L^{-1}[\Phi(s)]$

$$\therefore\quad L^{-1}\left[\frac{-2s}{(s^2+4)^2}\right] = -\frac{t}{2}\sin 2t \qquad \therefore\quad L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \frac{1}{4}t\sin 2t$$

$$\therefore\quad L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right] = \frac{1}{4}e^{-3t}\cdot t\sin 2t.$$

Alternatively we can use the above result (A).

$$L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right] = L^{-1}\left[\frac{s+3}{(s+3)^2+2^2}\right]$$

$$= e^{-3t}\cdot L^{-1}\left[\frac{s}{(s^2+2^2)^2}\right]$$

$$= e^{-3t}\cdot\frac{t}{2\cdot 2}\sin 2t \quad [\text{ Putting } a = 2 \text{ in the Ex. (1) }]$$

$$= \frac{1}{4}e^{-3t}\cdot t\cdot\sin 2t$$

Ex. 2 : Find the inverse Laplace transform of

(i) $\frac{s+1}{(s^2+2s-15)^2}$ (ii) $\frac{s^2+5}{(s^2+4s+13)^2}$

Sol. : (i) $L^{-1}\left[\frac{s+1}{(s^2+2s-15)^2}\right] = L^{-1}\left\{\frac{s+1}{[(s+1)^2-4^2]^2}\right\}$

$$= e^{-t}\cdot L^{-1}\left[\frac{s}{(s^2-4^2)^2}\right]$$

$$= e^{-t}\cdot t\cdot L^{-1}\int_s^\infty \frac{s}{(s^2-4^2)^2}\,ds$$

To find the integral put $s^2-4^2 = x$.

$$\therefore\quad \int\frac{s}{(s^2-4^2)^2}\,ds = \frac{1}{2}\int\frac{dx}{x^2} = -\frac{1}{2}\cdot\frac{1}{x} = -\frac{1}{2}\cdot\frac{1}{s^2-4^2}$$

$$\therefore \quad L^{-1}\left\{\frac{s+1}{(s^2+2s-15)^2}\right\} = e^{-t}\cdot t\cdot\frac{1}{2}\cdot L^{-1}\left[\frac{1}{s^2-4^2}\right]$$

$$= \frac{1}{2}e^{-t}\cdot t\cdot\frac{1}{4}\sin h\,4t = \frac{1}{8}t\,e^{-t}\sin h\,4t.$$

(ii) $$L^{-1}\left[\frac{s^2+5}{(s^2+4s+13)^2}\right] = L^{-1}\left[\frac{s^2+4s+13-(4s+8)}{(s^2+4s+13)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s^2+4s+13}\right] - 4\cdot L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right] - 4\cdot L^{-1}\left[\frac{s+2}{[(s+2)^2+3^2]^2}\right]$$

$$= e^{-2t}\cdot L^{-1}\left[\frac{1}{s^2+3^2}\right] - 4\cdot e^{-2t}\cdot L^{-1}\left[\frac{s}{(s^2+3^2)^2}\right]$$

$$= e^{-2t}\cdot\frac{1}{3}\sin 3t - 4\cdot e^{-2t}\frac{t}{2\cdot 3}\sin 3t$$

[By putting $a = 3$ in Ex. 1 (i)]

$$= \frac{e^{-2t}}{3}(1-2t)\sin 3t.$$

EXERCISE

Find the inverse Laplace transform of,

1. $\frac{s}{(s^2+16)^2}$ 2. $\frac{s+2}{(s^2+4s+5)^2}$ 3. $\frac{s}{(s^2-a^2)^2}$

4. $\frac{s-1}{(s^2-2s+2)^2}$ 5. $\frac{s+2}{(s^2+4s+3)^2}$ 6. $\frac{4(s-1)}{(s^2-2s+5)^2}$

7. $\frac{s+1}{(s^2+2s+2)^2}$

[**Ans.:** (1) $\frac{1}{8}t\sin 4t$, (2) $\frac{1}{2}te^{-2t}\sin t$, (3) $\frac{1}{2a}t\sin h\,at$,

(4) $\frac{1}{2}t\,e^{t}\sin t$, (5) $\frac{1}{2}t\,e^{-2t}\sin ht$, (6) $e^{t}\,t\sin 2t$, (7) $\frac{1}{2}t\,e^{-t}\sin t$]

Ex. 1 : Using Convolution theorem, prove that

$$\int_0^t \sin u\cos(t-u)\,du = \frac{1}{2}t\sin t.$$

Sol. : Let $F(t) = \int_0^t \sin u\cos(t-u)\,du$ and note that $F(t)$ is convolution of $f_1(t) = \sin t$ sin $f_2(t) = \cos t$.

By Convolution theorem (Note 1 page 4-19) Laplace transform of the convolution is equal to the product of the Laplace transforms of the two functions.

$$\therefore \quad L\,F(t) = L\sin t\cdot L\cos t$$

$$= \frac{1}{s^2+1}\cdot\frac{s}{s^2+1} = \frac{s}{(s^2+1)^2}$$

$$\therefore \quad F(t) = L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = L^{-1}\left[-\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

$$= -\frac{1}{2}t(-1)\,L^{-1}\left[\frac{1}{s^2+1}\right] = \frac{1}{2}t\cdot\sin t$$

(By the result (f) on page 4-31)

Ex. 2 : Using convolution theorem, prove that

$$\int_0^t J_o(u)\cdot J_o(t-u)\,du = \sin t,\ \text{ where } L\,J_o(t) = \frac{1}{\sqrt{s^2+1}}.$$

Sol. : Let $F(t) = \int_0^t J_o(u)\cdot J_o(t-u)\,du$ and note that $F(t)$ is the convolution of $f_1(t) = J_0(t)$ and $f_2(t) = J_0(t)$.

By convolution theorem

$$\therefore \quad L\,F(t) = L\,J_o(t)\cdot L\,J_o(t) = \frac{1}{\sqrt{s^2+1}}\cdot\frac{1}{\sqrt{s^2+1}} = \frac{2}{s^2+1}$$

$$\therefore \quad F(t) = L^{-1}\frac{1}{s^2+1} = \sin t$$

$$\therefore \quad \int_0^t J_o(t)\cdot J_o(t-u)\,du = \sin t.$$

EXERCISE

Using convolution theorem evaluate.

1. $\int_0^t u^2\cdot e^{-a(t-u)}\,du$ 2. $\int_0^t \cos u\cdot\sin(t-u)\,du$

3. $\int_0^t \sin u\cdot e^{-a(t-u)}\,du$ 4. $\int_0^t \cos u\cdot e^{-a(t-u)}\,du$

[**Ans.:** (1) $\frac{2}{a^3}\left(1-a\,t+a^2\frac{t^2}{2}-e^{-at}\right)$ (2) $\frac{1}{2}t\sin t$

(3) $\frac{1}{a^2+1}(a\sin t-\cos t+e^{-at})$ (4) $\frac{1}{a^2+1}(a\cos t+\sin t-e^{-at})$]

Ex. 1 : Using convolution theorem, prove that

(i) $L^{-1}\left[\frac{1}{s}\log\left(a+\frac{b}{s^2}\right)\right] = \int_0^t \frac{2}{u}\left[1-\cos\left(\frac{b}{a}\right)u\right]du.$ **(M.U. 1994)**

(ii) $L^{-1}\left[\frac{1}{s}\log\left(\frac{s+a}{s+b}\right)\right] = \int_0^t \frac{e^{-bu}-e^{-au}}{u}\,du.$ **(M.U. 2004)**

Sol. : (i) Let $\Phi_1(s) = \log\left(a+\frac{b}{s^2}\right)$ and let $\Phi_2(s) = \frac{1}{s}$.

$\therefore \quad \Phi_1(s) = \log(a^2+b) - \log s^2$

$\therefore \quad \Phi_1'(s) = \frac{2as}{as^2+b} - \frac{2}{s}$

$\therefore \quad L^{-1}\Phi_1'(s) = 2L^{-1}\frac{s}{s^2+(b/a)} - 2L^{-1}\frac{1}{s} = 2\cos\left(\frac{b}{a}\right)t - 2$

$\therefore$ By the theorem proved in (f) on page 4-31.

$$L^{-1}[\Phi_1(s)] = -\frac{1}{t}L^{-1}[\Phi_1'(s)]$$

$\therefore \quad L^{-1}\Phi_1(s) = -\frac{1}{t}\cdot 2\left[\cos\left(\frac{b}{a}\right)t-1\right] = \frac{2}{t}\left[1-\cos\left(\frac{b}{a}\right)t\right]$

Now $\Phi_2(s) = \frac{1}{s} \quad \therefore \quad L^{-1}\Phi_2(s) = 1$

$\therefore$ By the corollary of convolution theorem (page 4-19)

$$L^{-1}\Phi(s) = \int_0^t \frac{2}{u}\left[1-\cos\left(\frac{b}{a}\right)u\right]\cdot 1\cdot du$$

(ii) Let $\Phi_1(s) = \log\left(\frac{s+a}{s+b}\right)$ and $\Phi_2(s) = \frac{1}{s}$

$\therefore \quad \Phi_1(s) = \log(s+a) - \log(s+b)$

$\therefore \quad \Phi_1'(s) = \frac{1}{s+a} - \frac{1}{s+b}$

$\therefore \quad L^{-1}\Phi_1'(s) = L^{-1}\left(\frac{1}{s+a}\right) - L^{-1}\left(\frac{1}{s+b}\right) = e^{-at} - e^{-bt}$

$\therefore$ By the theorem proved in (f) on page 4-19.

$$L^{-1}[\Phi_1(s)] = -\frac{1}{t}L^{-1}[\Phi_1'(s)]$$

$\therefore \quad L^{-1}\Phi_1(s) = -\frac{1}{t}\left(e^{-at}-e^{-bt}\right) = \frac{1}{t}\left(e^{-bt}-e^{-at}\right)$

Now $\Phi_2(s) = \frac{1}{s} \quad \therefore \quad L^{-1}\Phi_2(s) = 1$

$\therefore$ By the corollary of convolution theorem (page 4-19)

$$L^{-1}\Phi(s) = \int_0^t \frac{1}{u}\left(e^{-bu}-e^{-au}\right)\cdot 1\cdot du.$$

[See ex. 4(v) on page 4-28.]

Ex. 2 : Using convolution theorem, prove that

(i) $L^{-1}\left[\frac{1}{s}\tan^{-1}\frac{2}{s}\right] = \int_0^t \frac{1}{u}\cdot\sin 2u\,du$

(ii) $L^{-1}\left[\frac{1}{s}\cot^{-1}(s+1)\right] = \int_0^t \frac{1}{u}\cdot e^{-u}\sin u\,du$

Sol. : (i) Let $\Phi_1(s) = \tan^{-1}\frac{2}{s}$ and $\Phi_2(s) = \frac{1}{s}$

$\therefore \quad \Phi_1'(s) = \frac{1}{1+(4/s^2)}\left(-\frac{2}{s^2}\right) = -\frac{2}{s^2+4}$

$\therefore \quad L^{-1}\Phi_1'(s) = -\sin 2t$

$\therefore$ By the theorem proved in (f) on page 4-31.

$$L^{-1}[\Phi_1(s)] = -\frac{1}{t}L^{-1}[\Phi_1'(s)]$$

$\therefore \quad L^{-1}\Phi_1(s) = \frac{1}{t}\sin 2t$

Now $\Phi_2(s) = \frac{1}{s} \quad \therefore \quad L^{-1}\Phi_2(s) = 1$

$\therefore$ By the corollary of convolution theorem

$$L^{-1}\Phi(s) = \int_0^t \frac{1}{u}\sin 2u\cdot 1\cdot du$$

(ii) Let $\Phi_1(s) = \cot^{-1}(s+1)$, $\Phi_2(s) = \frac{1}{s}$

$\therefore \quad \Phi_1(s) = \cot^{-1}(s+1) = \frac{\pi}{2} - \tan^{-1}(s+1)$

$\therefore \quad \Phi'(s) = -\frac{1}{1+(s+1)^2}$

$\therefore \quad L^{-1}\Phi'(s) = -L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$

$= -e^{-t}\left[\frac{1}{s^2+1}\right] = -e^{-t}\sin t$ (By shifting theorem)

$\therefore$ By the theorem proved in (f) on page 4-31.

$$L^{-1}[\Phi_1(s)] = -\frac{1}{t} L^{-1}[\Phi(s)]$$

$$\therefore \quad L^{-1}\Phi_1(s) = \frac{1}{t} \cdot e^{-t} \sin t$$

Now $\Phi_2(s) = \frac{1}{s}$ $\quad \therefore \quad L^{-1}\Phi_2(s) = 1$

$\therefore$ By the corollary of convolution theorem

$$L^{-1}\Phi(s) = \int_0^t \frac{1}{u} e^{-u} \sin u \, du.$$

EXERCISE

Using convolution theorem prove that,

1. $L^{-1}\left[\frac{1}{s}\log\left(\frac{s+3}{s+4}\right)\right] = \int_0^t \frac{e^{-4u} - e^{-3u}}{u} du.$

2. $L^{-1}\left[\frac{1}{s}\tan^{-1}\frac{1}{s}\right] = \int_0^t \frac{1}{u} \cdot \sin u \, du.$

3. $L^{-1} = \left[\frac{1}{s}\cot^{-1} s\right] = \int_0^t \frac{1}{u} \cdot \sin u \, du.$

4. $L^{-1}\left[\frac{1}{s} \cdot \log\left(\frac{s+1}{s+2}\right)\right] = \int_0^t \left(\frac{e^{-2u} - e^{-u}}{u}\right) du$ **(M.U. 2004)**

5. $L^{-1}\left[\frac{1}{s} \cdot \log\left(\frac{s+2}{s+3}\right)\right] = \int_0^t \left(\frac{e^{-3u} - e^{-2u}}{u}\right) du$ **(M.U. 2001)**

6. $L^{-1}\left[\frac{1}{s}\log\left(1 + \frac{2}{s^2}\right)\right] = \int_0^t \frac{2}{u}[1 - \cos 2u] \, du$

(g) Use of differentiation of $f(t)$: From § 10, page 3-50, if $f(0) = f'(0) = f''(0) = \ldots\ldots 0$, we have $L\,f'(t) = s\,L[f(t)]$. If we write,

$L\,f(t) = \Phi(s)$, then $L\,f'(t) = s\,\Phi(s)$. Hence, $f'(t) = L^{-1}[s\,\Phi(s)]$.

Similarly, $L^{-1}[s^2\Phi(s)] = f''(t)$.

Thus, we have

$$\boxed{L^{-1}[s\,\Phi(s)] = \frac{d}{dt} f(t),\ L^{-1}[s^2\,\Phi(s)] = \frac{d^2}{dt^2} f(t), \ldots\ldots L^{-1}[s^n\Phi(s)] = \frac{d^n}{dt^n} f(t)} \qquad \ldots\ldots\ldots (18)$$

Ex. 1 : Find the inverse Laplace transform of

(i) $\frac{s^2}{(s^2 + a^2)^2}$ (ii) $\frac{s^2}{(s+a)^3}$

Sol. : (i) Let $\Phi(s) = \frac{1}{s^2 + a^2}$ $\quad \therefore \quad \Phi'(s) = -\frac{2s}{(s^2 + a^2)^2}$

But $L^{-1}\,\Phi(s) = \frac{1}{a}\sin at$ and $L^{-1}[\Phi'(s)] = -t\,L^{-1}\Phi(s)$

$$\therefore \quad L^{-1}\left[-\frac{2s}{(s^2 + a^2)^2}\right] = -t \cdot \frac{1}{a}\sin at$$

$$\therefore \quad L^{-1}\left[\frac{s}{(s^2 + a^2)^2}\right] = \frac{1}{2a} \cdot t \sin at$$

But $L^{-1}[s \cdot \Phi(s)] = \frac{d}{dt}[L^{-1}\Phi(s)]$

$$\therefore \quad L^{-1}\left[s \cdot \frac{s}{(s^2 + a^2)^2}\right] = \frac{d}{dt}\left[\frac{1}{2a} \cdot t \sin at\right]$$

$$= \frac{1}{2a}[\sin at + at \cos at]$$

[See Ex. 1(i), on page 4-20]

(ii) Let $\Phi(s) = \frac{1}{s+a}$

Now, $L^{-1}\frac{1}{(s+a)^3} = e^{-at} L^{-1}\frac{1}{s^3} = e^{-at}\frac{t^2}{2}$ (by shifting theorem)

But $L^{-1}[s^2\,\Phi(s)] = \frac{d^2}{dt^2}\left[L^{-1}\Phi(s)\right]$

$$\therefore \quad L^{-1}\left[s^2 \cdot \frac{1}{(s+a)^3}\right] = \frac{d^2}{dt^2}\left[\frac{1}{2} \cdot t^2 \cdot e^{-at}\right] = \frac{1}{2}\left[a^2 t^2 - 4at + 2\right]e^{-at}$$

Ex. 2 : Find the inverse Laplace transform of $\frac{(s-1)^2}{(s^2 - 2s + 5)^2}$.

Sol. : $L^{-1}\frac{(s-1)^2}{(s^2 - 2s + 5)^2} = L^{-1}\frac{(s-1)^2}{[(s-1)^2 + 2^2]^2} = e^t\,L^{-1}\frac{s^2}{(s^2 + 2^2)^2}$

Let $\Phi(s) = \frac{1}{(s^2 + 2^2)}$ $\quad \therefore \quad \Phi'(s) = -\frac{2s}{(s^2 + 2^2)^2}$

$$L^{-1}\Phi(s) = \frac{1}{2} \cdot \sin 2t$$

$$\therefore \quad L^{-1}\Phi'(s) = -t\,L^{-1}\Phi(s) = -\frac{1}{2}t\sin 2t$$

$$\therefore \quad L^{-1}\frac{s}{(s^2+2^2)^2} = \frac{1}{2\cdot 2}\cdot t\sin 2t$$

$$\therefore \quad L\left\{s\cdot\frac{s}{(s^2+2^2)^2}\right\} = \frac{d}{dt}\left\{\frac{1}{4}t\sin 2t\right\}$$

$$= \frac{1}{4}[\sin 2t + 2t\cos 2t]$$

$$\therefore \quad L^{-1}\frac{(s-1)^2}{(s^2-2s+5)^2} = \frac{1}{4}e^{t}\cdot[\sin 2t + 2t\cos 2t]$$

EXERCISE

Find the inverse Laplace transform of,

1. $\dfrac{s^2}{(s+1)^3}$ 2. $\dfrac{s^2}{(s^2-a^2)^2}$ 3. $\dfrac{s^2}{(s+4)^3}$ 4. $\dfrac{s^2}{(s^2+1)^2}$

5. $\dfrac{s^2}{(s^2-1)^2}$ 6. $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$

[**Ans. :** (1) $\frac{1}{2}(t^2-4t+12)e^{-t}$, (2) $\frac{1}{2a}[\sin h\,at + at\cos h\,at]$,

(3) $(8t^2-4t+1)e^{-4t}$, (4) $\frac{1}{2}[\sin t + t\cos t]$,

(5) $\frac{1}{2}[\sin ht + t\cos ht]$,

(6) **Hint :** $L^{-1}\dfrac{(s+2)^2}{[(s+2)^2+2^2]^2} = e^{-2t}L^{-1}\dfrac{s^2}{(s^2+2^2)}$

Ans. : $\frac{1}{4}e^{-2t}(2t\cos 2t + \sin 2t)$.]

(h) Use of integration $f(t)$ (Division by s) : We have proved in § 11 of the last chapter (page 3-52) that if $L\,f(t) = \Phi(s)$ then $L\int_0^t f(u)\,du = \frac{1}{s}\Phi(s)$.

Taking the inverse Laplace transform of both sides.

$$\int_0^t f(u)\,du = L^{-1}\frac{1}{s}\Phi(s)$$

But $f(u) = L^{-1}\Phi(s)$

$$\therefore \quad \boxed{L^{-1}\frac{1}{s}\Phi(s) = \int_0^t L^{-1}\Phi(s)\,ds} \qquad \text{.........(19)}$$

Ex. 1 : Find : (i) $L^{-1}\left[\dfrac{1}{s(s^2+4)}\right]$ (ii) $L^{-1}\dfrac{1}{s^2(s+1)}$

(iii) $L^{-1}\dfrac{1}{s^3(s^2+a^2)}$ **(M.U. 1993)** (iv) $L^{-1}\dfrac{s^2-a^2-s^3}{s^3(s^2-a^2)}$.

(v) $L^{-1}\dfrac{1}{s\sqrt{s+4}}$ **(M.U. 2002)**

Sol. : (i) $L^{-1}\left[\dfrac{1}{s(s^2+4)}\right] = \int_0^t L^{-1}\left[\dfrac{1}{(s^2+4)}\right]\cdot du$

$$= \int_0^t \frac{1}{2}\cdot\sin 2u\,du = \frac{1}{2}\left[\frac{-\cos 2u}{2}\right]_0^t$$

$$= \frac{1}{4}[1-\cos 2t] \quad \text{[See Ex. 6, page 4-30]}$$

(ii) $L^{-1}\left[\dfrac{1}{s^2(s+1)}\right] = L^{-1}\left[\dfrac{1}{s}\cdot\dfrac{1}{s(s+1)}\right] = \int_0^t L^{-1}\left[\dfrac{1}{s(s+1)}\right]\cdot du$

$$= \int_0^t L^{-1}\left[\frac{1}{s}-\frac{1}{s+1}\right]du = \int_0^t L^{-1}\left[\frac{1}{s}\right]du - \int_0^t L^{-1}\left[\frac{1}{s+1}\right]du$$

$$= \int_0^t 1\cdot dt - \int_0^t e^{-u}\,du = [\,u\,]_0^t + \left[e^{-u}\right]_0^t$$

$$= t + e^{-t} - 1. \quad \text{[See Ex. 8, page 4-15]}$$

Aliter : We can use (19) repeatedly.

$$\therefore \quad L^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{s+1}\right] = \int_0^t L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s+1}\right]du$$

$$= \int_0^t\int_0^t L^{-1}\left(\frac{1}{s+1}\right)(du)^2 = \int_0^t\int_0^t (e^{-u})(du)^2$$

$$= \int_0^t\left[-e^{-u}\right]_0^1 du = \int_0^t (1-e^{-u})\,du$$

$$= \left[u+e^{-u}\right]_0^t = t + e^{-t} - 1.$$

(iii) $L^{-1}\left[\dfrac{1}{s^3(s^2+a^2)}\right] = L^{-1}\left[\dfrac{1}{s}\cdot\dfrac{1}{s^2(s^2+a^2)}\right]$

$$= \int_0^t L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]du = \frac{1}{a^2}\int_0^t L^{-1}\left[\frac{1}{s^2}-\frac{1}{s^2+a^2}\right]du$$

$$= \frac{1}{a^2}\int_0^t \left[u - \frac{1}{a}\sin au\right] du$$

$$= \frac{1}{a^2}\left[\left\{\frac{u^2}{2}\right\}_0^t + \frac{1}{a^2}\{\cos au\}_0^t\right]$$

$$= \frac{1}{a^2}\left[\frac{t^2}{2}\right] + \frac{1}{a^4}[\cos at - 1]$$

Aliter : We may also use (19) repeatedly.

$$\therefore\ L^{-1}\left[\frac{1}{s^3}\cdot\frac{1}{(s^2+a^2)}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2(s^2+a^2)}\right]$$

$$= \int_0^t L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] du = \int_0^t L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s(s^2+a^2)}\right] du$$

$$= \int_0^t\int_0^t L^{-1}\left[\frac{1}{s(s^2+a^2)}\right](du)^2$$

$$= \int_0^t\int_0^t\int_0^t L^{-1}\frac{1}{(s^2+a^2)}(du)^3 = \int_0^t\int_0^t\int_0^t \frac{1}{a}\sin(au)\,(du)^3$$

$$= \frac{1}{a}\int_0^t\int_0^t\left[-\frac{\cos au}{a}\right]_0^t (du)^2 = \frac{1}{a^2}\int_0^t\int_0^t (1-\cos au)\,(du)^2$$

$$= \frac{1}{a^2}\int_0^t\left[u - \frac{\sin au}{a}\right]_0^t du = \frac{1}{a^2}\int_0^t\left(u - \frac{\sin au}{a}\right) du$$

$$= \frac{1}{a^2}\left[\frac{u^2}{2} + \frac{\cos au}{a^2}\right]_0^t = \frac{1}{a^2}\left[\frac{t^2}{2} + \frac{\cos at}{a^2} - \frac{1}{a^2}\right]$$

$$= \frac{1}{a^2}\left[\frac{t^2}{2}\right] + \frac{1}{a^4}(\cos at - 1)$$

(iv) $$L^{-1}\frac{s^2-a^2-s^3}{s^3(s^2-a^2)} = L^{-1}\left[\frac{1}{s}\cdot\frac{s^2-a^2-s^3}{s^2(s^2-a^2)}\right]$$

$$= \int_0^t L^{-1}\left[\frac{s^2-a^2-s^3}{s^2(s^2-a^2)}\right] du = \int_0^t L^{-1}\left[\frac{1}{s^2} - \frac{s}{s^2-a^2}\right] du$$

$$= \int_0^t u\,du - \int_0^t \cos h\,au\,du = \left[\frac{u^2}{2}\right]_0^t - \left[\frac{\sin h\,au}{a}\right]_0^t$$

$$= \frac{t^2}{2} - \frac{1}{a}\sin h\,at.$$

(v) $$L^{-1}\frac{1}{s\sqrt{s+4}} = \int_0^t L^{-1}\frac{1}{\sqrt{s+4}}\,du$$ [See Ex. 4 (iv), page 4-27]

$$= \int_0^t e^{-4u} L^{-1}\left(\frac{1}{\sqrt{s}}\right) du = \int_0^t e^{-4u}\cdot\frac{u^{-1/2}}{\overline{|1/2}}\,du$$

$$= \frac{1}{\sqrt{\pi}}\int_0^t e^{-4u} u^{-1/2}\,du$$

Now put $4u = x^2$, $2du = x\,dx$, $\sqrt{u} = \frac{x}{2}$.

$$\therefore\ L^{-1}\frac{1}{s\sqrt{s+4}} = \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-x^2}\cdot\frac{2}{x}\cdot\frac{x}{2}\,dx$$

$$= \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-x^2} dx = \frac{1}{2}\cdot\frac{2}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-x^2} du$$

$$= \frac{1}{2}\,erf(2\sqrt{t}) \quad \left[\because \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2} du = erf(t)\right]$$

EXERCISE

Find : **1.** $L^{-1}\left[\frac{1}{s(s^2+9)}\right]$ **2.** $L^{-1}\left[\frac{1}{s^2(s+2)}\right]$

3. $L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]$ **(M.U. 1994)** **4.** $L^{-1}\frac{1}{s(s^2-a^2)}$

5. $\frac{54}{s^3(s-3)}$ **(M.U. 2003)**

[**Ans.: (1)** $\frac{1}{9}[1-\cos 3t]$, **(2)** $\frac{1}{4}[2t + e^{-2t} - 1]$,

(3) $\frac{t^2}{2} + \cos t - 1$, **(4)** $\frac{1}{a^2}[\cos h\,at - 1]$,

(5) $[-2 - 6t - 9t^2 + 2e^{3t}]$]

Miscellaneous Examples

Ex. 1 : Find $\int_0^\infty \cos(tx^2)\,dx$ and hence, find $\int_0^\infty \cos x^2\cdot dx$.

Sol. : Let $f(t) = \int_0^\infty \cos(tx^2)\,dx$

$$\therefore \quad L f(t) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} \int_0^\infty [\cos(tx^2)\, dx]\, dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} \cos(tx^2)\, dt\right] dx$$

$$= \int_0^\infty [L \cos(tx^2)]\, dx = \int_0^\infty \frac{s}{s^2 + x^4}\, dx$$

Now put $x = \sqrt{s \tan\theta}$ $\quad \therefore \quad dx = \dfrac{s \cdot \sec^2\theta\, d\theta}{2\sqrt{s \tan\theta}}$

$$\therefore \quad L[f(t)] = \int_0^{\pi/2} \frac{s}{s^2 + s^2 \tan^2\theta} \cdot \frac{s \cdot \sec^2\theta\, d\theta}{2\sqrt{s\tan\theta}}$$

$$= \int_0^{\pi/2} \frac{1}{2\sqrt{s\tan\theta}}\, d\theta = \frac{1}{2\sqrt{s}} \int_0^{\pi/2} (\sin\theta)^{-1/2} (\cos\theta)^{1/2}\, d\theta$$

$$= \frac{1}{2\sqrt{s}} \frac{\Gamma(1/4)\,\Gamma(3/4)}{2 \cdot 1!} \left[\int_0^{\pi/2} \sin^p\theta \cos^p\theta\, d\theta = \frac{\Gamma((p+1)/2)\;\Gamma((q+1)/2)}{2\Gamma((p+q+2)/2)}\right]$$

$$= \frac{1}{2\sqrt{s}} \cdot \frac{\sqrt{2}\cdot\pi}{2} \qquad \left[\because \Gamma(1/4)\;\Gamma(3/4) = \sqrt{2}\cdot\pi\right]$$

$$\therefore \quad L(f(t) = \frac{\pi}{2\sqrt{2}\sqrt{s}}$$

$$\therefore \quad f(t) = \frac{\pi}{2\sqrt{2}} L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{\pi}{2\sqrt{2}} \cdot \frac{t^{-1/2}}{\Gamma(1/2)} = \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{1}{\sqrt{t}}$$

Now put $t = 1$, $\quad \therefore \quad \int_0^\infty \cos x^2 dx = \dfrac{\sqrt{\pi}}{2\sqrt{2}} = \dfrac{1}{2}\sqrt{\dfrac{\pi}{2}}$.

Ex. 2 : Find $\int_0^\infty \sin(tx^2)$ and hence, find $\int_0^\infty \sin x^2\, dx$. **(M.U. 2003)**

Sol. : Let $f(t) = \int_0^\infty \sin(tx^2)\, dx$

$$\therefore \quad L f(t) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-st} \left[\int_0^\infty \sin(tx^2)\, dx\right] dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} \sin(tx^2)\, dt\right] dx$$

$$= \int_0^\infty [L \sin(tx^2)]\, dx = \int_0^\infty \frac{x^2}{s^2 + x^4}\, dx$$

Now put $x = \sqrt{s\tan\theta}$, $dx = \dfrac{s \cdot \sec^2\theta\, d\theta}{2\sqrt{s\tan\theta}}$

$$\therefore \quad L f(t) = \int_0^{\pi/2} \frac{s \tan\theta}{s^2 + s^2\tan^2\theta} \cdot \frac{s \cdot \sec^2\theta}{2\sqrt{s\tan\theta}} \cdot d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$$= \frac{1}{2\sqrt{s}} \frac{\Gamma(3/4)\;\Gamma(1/4)}{2 \cdot 1} = \frac{1}{2\sqrt{s}} \frac{\sqrt{2}\cdot\pi}{2}$$

$$= \frac{\pi}{2\sqrt{2}} \cdot \frac{1}{\sqrt{s}}$$

$$\therefore \quad f(t) = \frac{\pi}{2\sqrt{2}} L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{\pi}{2\sqrt{2}} \cdot \frac{t^{-1/2}}{\Gamma(1/2)} = \frac{\sqrt{\pi}}{2\sqrt{2}} \cdot \frac{1}{\sqrt{t}}$$

Now put $t = 1$, $\quad \therefore \quad \int_0^\infty \sin x^2 dx = \dfrac{\sqrt{\pi}}{2\sqrt{2}} = \dfrac{1}{2}\sqrt{\dfrac{\pi}{2}}$.

Ex. 3 : Find $\int_0^\infty e^{-tx^2} dx$ and hence, find $\int_0^\infty e^{-x^2} dx$.

Sol. : Let $f(t) = \int_0^\infty e^{-tx^2} dx$

$$\therefore \quad L f(t) = \int_0^\infty e^{-st}\left[\int_0^\infty e^{-tx^2} dx\right] dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} \cdot e^{-tx^2} dt\right] dx$$

$$= \int_0^\infty \left[L(e^{-tx^2})\right] dx = \int_0^\infty \frac{dx}{s + x^2} \qquad \left[\because L\, e^{-at} = \frac{1}{s+a}\right]$$

$$= \left[\frac{1}{\sqrt{s}} \tan^{-1} \frac{x}{\sqrt{s}}\right]_0^\infty = \frac{\pi}{2\sqrt{s}}$$

$$\therefore \quad f(t) = \frac{\pi}{2} L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{\pi}{2} \frac{t^{-1/2}}{\Gamma(1/2)}$$

$$= \frac{\pi}{2} \frac{1}{\sqrt{\pi}\sqrt{t}} = \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

Now, put $t = 1$, $\quad \therefore \quad \int_0^\infty e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$.

Ex. 4 : If $J_0(t) = \dfrac{1}{\pi}\int_0^\pi \cos(t\cos\theta)\, d\theta$, prove that $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$.

Sol. : We have $J_0(t) = \dfrac{1}{\pi}\int_0^{\pi} \cos(t\cos\theta)\,d\theta = \dfrac{2}{\pi}\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta$

Taking Laplace transforms of both sides.

$$L[J_0(t)] = \frac{2}{\pi}\int_0^{\infty} e^{-st}\left[\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta\right]dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left[\int_0^{\infty} e^{-st}\cos(t\cos\theta)\,dt\right]d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left[L\cos(t\cos\theta)\right]d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left[\frac{s}{s^2+\cos^2\theta}\right]d\theta = \frac{2}{\pi}\int_0^{\pi/2}\frac{s\sec^2\theta}{s^2\sec^2\theta+1}\,d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\frac{s\sec^2\theta}{(s^2+1)+s^2\tan^2\theta}\,d\theta$$

Put $s\tan\theta = t$ $\quad\therefore\ s\sec^2\theta\,d\theta = dt$

$$\therefore\ L[J_0(t)] = \frac{2}{\pi}\int_0^{\infty}\frac{dt}{t^2+(s^2+1)}$$

$$= \frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\left[\tan^{-1}\left(\frac{t}{\sqrt{s^2+1}}\right)\right]_0^{\infty}$$

$$= \frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\cdot\frac{\pi}{2} = \frac{1}{\sqrt{s^2+1}}.$$

Ex. 5 : If $J_0(t) = \dfrac{1}{\pi}\int_0^{\infty}\cos(t\sin\theta)\,d\theta$, prove that $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$. **(M.U. 2003)**

Sol. : Do it yourself.

Ex. 6 : Prove that $L^{-1}\left[\dfrac{1}{s}\cos\dfrac{1}{s}\right] = 1 - \dfrac{t^2}{(2!)^2} + \dfrac{t^4}{(4!)^2} - \dfrac{t^6}{(6!)^2} + \ldots\ldots$

Sol. : We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$$

$$\therefore\ \cos\left(\frac{1}{s}\right) = 1 - \frac{1}{2!\,s^2} + \frac{1}{4!}\cdot\frac{1}{s^4} - \frac{1}{6!}\cdot\frac{1}{s^6} + \ldots\ldots$$

$$\therefore\ \frac{1}{s}\left\{\cos\frac{1}{s}\right\} = \frac{1}{s}\left\{1 - \frac{1}{2!\,s^2} + \frac{1}{4!}\cdot\frac{1}{s^4} - \frac{1}{6!}\cdot\frac{1}{s^6} + \ldots\ldots\right\}$$

$$\therefore\ L^{-1}\left\{\frac{1}{s}\cos\frac{1}{s}\right\} = L^{-1}\frac{1}{s}\left\{1 - \frac{1}{2!\,s^2} + \frac{1}{4!\,s^4} - \frac{1}{6!\,s^6} + \ldots\ldots\right\}$$

$$= L^{-1}\left\{1 - \frac{1}{2!\,s^3} + \frac{1}{4!\,s^5} - \frac{1}{6!\,s^7} + \ldots\ldots\right\}$$

$$= 1 - \frac{1}{2!}\cdot\frac{t^2}{2!} + \frac{1}{4!}\cdot\frac{t^4}{4!} - \frac{1}{6!}\cdot\frac{t^6}{6!} + \ldots\ldots$$

$$= 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots\ldots$$

4. Laplace Transform Of Periodic Functions

If $f(t)$ is a periodic function of period a, show that

$$\boxed{L\,f(t) = \frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)\,dt} \quad \textbf{(M.U. 1995, 96)} \quad \ldots\ldots\ldots (20)$$

Proof. : Since $f(t)$ is periodic with period a, $f(t) = f(t+a) = f(t+2a) = \ldots\ldots$

$$\therefore\ L\,f(t) = \int_0^{\infty} e^{-st}f(t)\,dt = \int_0^a e^{-st}f(t)\,dt + \int_a^{2a} e^{-st}f(t)\,dt + \ldots\ldots\infty$$

Now $\displaystyle\int_a^{2a} e^{-st}f(t)\,dt = \int_0^a e^{-s(u+a)}f(u+a)\,du$ [where $t = u+a$]

$$= e^{-as}\int_0^a e^{-su}f(u+a)\,du$$

$$= e^{-as}\int_0^a e^{-st}f(t+a)\,dt \quad \text{(changing } u \text{ to } t\text{)}$$

$$= e^{-as}\int_0^a e^{-st}f(t)\,dt \qquad [f(t+a) = f(t)]$$

Similarly, we can show that the other integrals are also equal to $e^{-2as}\int_0^a e^{-st}f(t)\,dt$ and so on.

$$\therefore\ L\,f(t) = \left(1 + e^{-as} + e^{-2as} + \ldots\ldots\infty\right)\int_0^a e^{-st}f(t)\,dt.$$

$$\therefore\ \boxed{L\,f(t) = \frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)\,dt} \qquad \left[\text{For a G.P. } s_\infty = \frac{a}{1-r}\right]$$

Ex. 1 : Find Laplace transform of

(i) $f(t) = K\dfrac{t}{T}$ for $0 < t < T$ and $f(t) = f(t+T)$.

(ii) $f(t) = 1$, for $0 \le t < a$ and $f(t) = -1$, $a < t < 2a$ and $f(t)$ is periodic with period $2a$. **(M.U. 1999, 2003)**

Sol. : (i) Since $f(t)$ is periodic with period T

$$L\,f(t) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\,f(t)\,dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\cdot\frac{Kt}{T}\,dt$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\int_a^T e^{-st}\cdot t\cdot dt$$

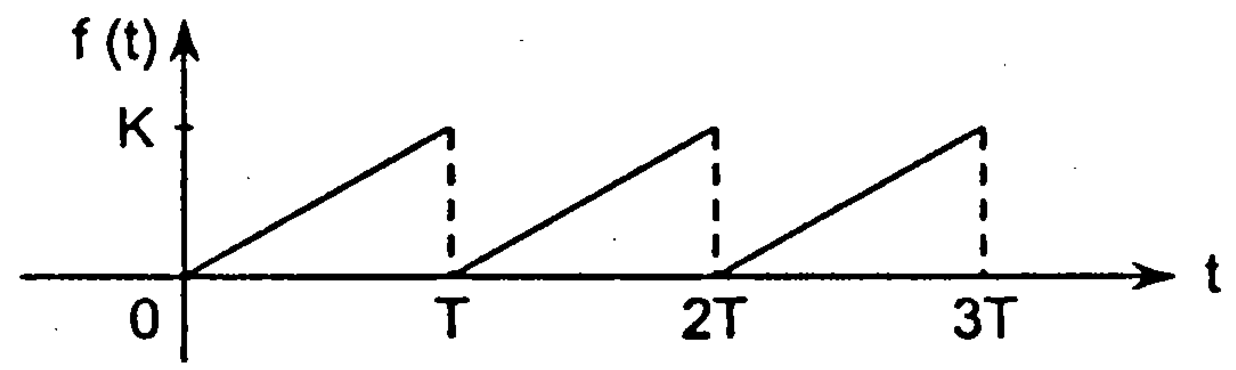
$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\left[-t\cdot\frac{e^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_0^T$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\left[-\frac{Te^{-sT}}{s}-\frac{e^{-st}}{s^2}+\frac{1}{s^2}\right]$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\left[\frac{1}{s^2}\left(1-e^{-sT}\right)-\frac{Te^{-sT}}{s}\right]$$

$$= K\left[\frac{1}{Ts^2}-\frac{e^{-sT}}{s(1-e^{-st})}\right]$$

The function $f(t) = k\cdot\frac{t}{T}$ is known as "saw-tooth wave" function and its graph is as shown below.



(ii) Since $f(t)$ is periodic with period $2a$,

$$L\,f(t) = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}\,f(t)\,dt$$

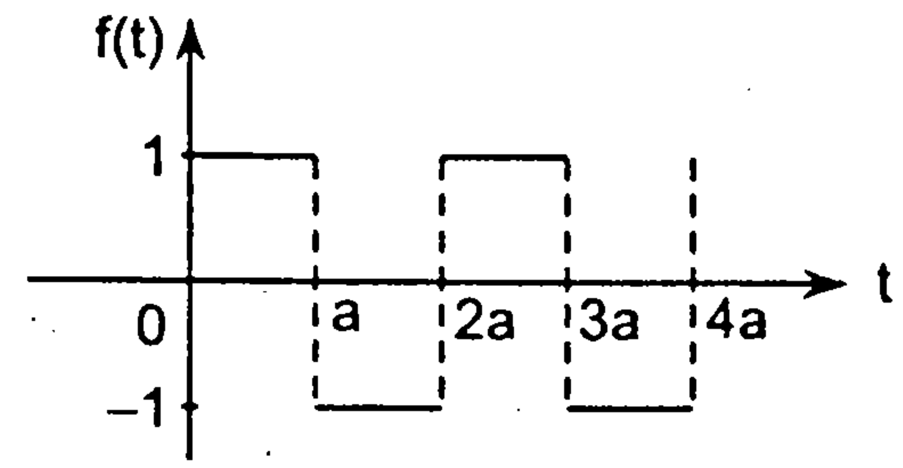
$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}(1)\,dt+\int_a^{2a} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left\{-\frac{e^{-st}}{s}\right\}_0^a+\left\{\frac{e^{-st}}{s}\right\}_a^{2a}\right]$$

$$= \frac{1}{s}\cdot\frac{1}{1-e^{-2as}}\cdot(1-e^{-as})^2 = \frac{1}{s}\cdot\frac{1-e^{-as}}{1+e^{-as}}$$

$$= \frac{1}{s}\cdot\left[\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+{}^{-as/2}}\right] = \frac{1}{s}\tan h\left\{\frac{as}{2}\right\}$$

The function is known as "square wave" function and its graph is shown below.



Ex. 2 : Find Laplace transform of $f(t) = a\sin pt,\ 0 < t < \pi/p,\ f(t) = 0,\ \pi/p < t < 2\pi/p$ and $f(t) = f(t+2\pi/p)$. **(M.U. 1995, 2002)**

Sol. : $L\,f(t) = \frac{1}{1-e^{-(2\pi/p)s}}\int_0^{\pi/p} e^{-st} a\sin pt\,dt$

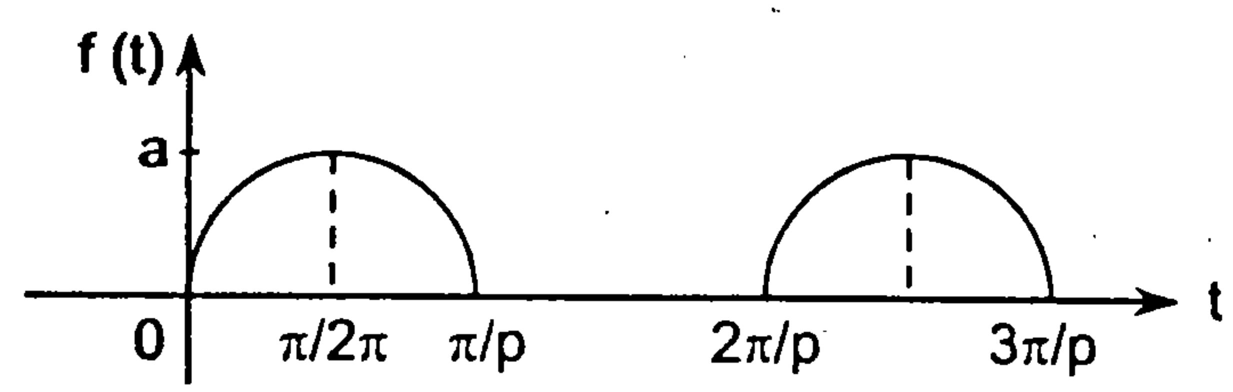
$$= \frac{a}{1-e^{-(2\pi/p)s}}\left[\frac{1}{s^2+p^2}\cdot e^{-st}(-s\sin pt-p\cos pt)\right]_0^{\pi/p}$$

$$= \frac{a}{1-e^{-(2\pi/p)s}}\cdot\frac{1}{(s^2+p^2)}\cdot\left[e^{-s\pi/p}(-s\sin\pi-p\cos\pi)\right.$$
$$\left.-e^0(-s\sin 0-p\cos 0)\right]$$

$$= \frac{a}{1-e^{-(2\pi/p)s}}\cdot\frac{1}{s^2+p^2}\left[p\,e^{-s\pi/p}+p\right]$$

$$= \frac{ap}{1-e^{-(s\pi/p)}}\cdot\frac{1}{s^2+p^2}$$

The function $f(t) = a\sin p\,t$ is known as "half-sine wave rectifier" function and its graph is shown below.



Ex. 3 : Find the Laplace transform of $f(t) = |\sin p\,t|,\ t \geq 0$. **(M.U. 2003)**

Sol. : We first note that

$$f\left(t+\frac{\pi}{p}\right) = \left|\sin p\left(t+\frac{\pi}{p}\right)\right|$$

$$= \left|\sin(pt+\pi)\right| = \left|\sin pt\right|$$

$\therefore$ $f(t)$ is a periodic function with period π/p.

Now, $L\,f(t) = \frac{1}{1-e^{-\pi s/p}}\int_0^{\pi/p} e^{-st}\,|\sin pt|\,dt$

$$= \frac{1}{1-e^{-\pi s/p}} \int_0^{\infty} e^{-st} \sin pt \, dt \quad \left[\because \sin pt > 0, \text{ for } 0 \le t \le \frac{\pi}{p}\right]$$

$$= \frac{1}{1-e^{-\pi s/p}} \left[\frac{e^{-st}}{s^2+p^2}(-s \sin pt - p \cos pt)\right]_0^{\pi/p}$$

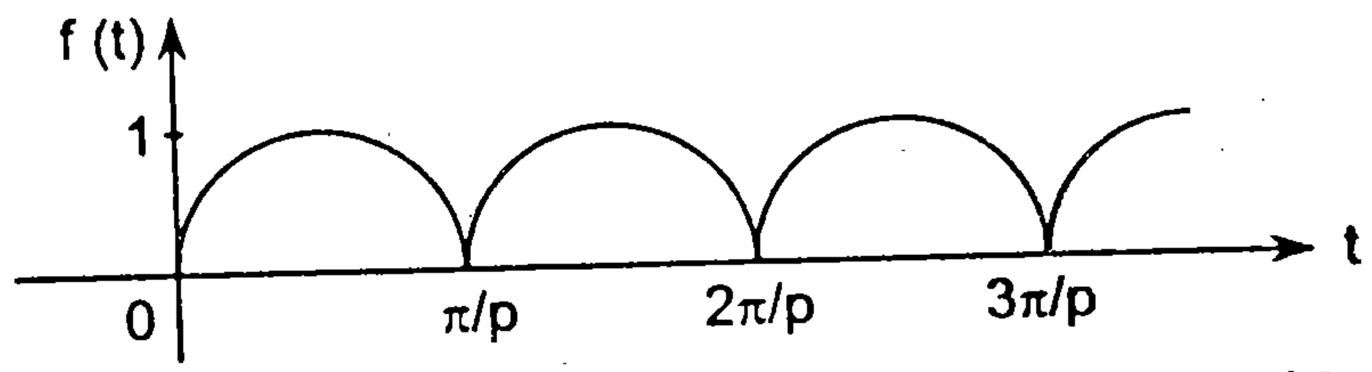
$$= \frac{1}{1-e^{-\pi s/p}} \cdot \frac{1}{(s^2+p^2)} \left[e^{-s\pi/p}(0+p) - (0-p)\right]$$

$$= \frac{1}{s^2+p^2} \cdot \frac{1}{1-e^{-\pi s/p}} \cdot p \cdot (1+e^{-\pi s/p})$$

$$= \frac{p}{s^2+p^2} \cdot \left(\frac{e^{\pi s/2p} + e^{-\pi s/2p}}{e^{\pi s/2p} - e^{-\pi s/2p}}\right)$$

$$= \frac{p}{s^2+p^2} \cdot \cot h\left(\frac{\pi s}{2p}\right)$$

The function $f(t) = |\sin pt|$ is known as "full-sine wave rectifier" and its graph is shown below.



Ex. 4 : Find Laplace transform of $f(t) = \sin 2t,\ 0 < t < \pi/2$,
$f(t) = 0, \pi/2 < t < \pi$ and $f(t) = f(t+\pi)$. **(M.U. 1995, 2002)**

Sol. : In the above Ex. 2, put $p = 2$, $a = 1$.

$$\therefore \quad L f(t) = \frac{1}{1-e^{-\pi/s}} \cdot \frac{1}{s^2+2^2} \cdot \left[2e^{-\pi s/2} + 2\right]$$

$$= \frac{2}{1-e^{-\pi s/2}} \cdot \frac{1}{s^2+2^2}.$$

Ex. 5 : Find $L f(t)$ where, $f(t) = t,\ 0 < t < 1$; $f(t) = 0,\ 1 < t < 2$ and
$f(t+2) = f(t)$ for $t > 0$. **(M.U. 1996)**

Sol. : Since $f(t)$ is periodic with period $a = 2$, we have

$$\therefore \quad L f(t) = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\, dt = \frac{1}{1-e^{-2s}} \int_0^a e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} \cdot t\, dt + \int_1^2 e^{-st} \cdot 0 \cdot dt\right]$$

$$= \frac{1}{1-e^{-2s}} \left[t\left(-\frac{e^{-st}}{s}\right) - \left(\frac{e^{-st}}{s^2}\right)(1)\right]_0^1$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{1}{s^2(1-e^{-2s})}(1 - e^{-s} - s\,e^{-s}).$$

EXERCISE

Find the Laplace transform of

1. $f(t) = \dfrac{t}{a},\ 0 < t \le a$; $f(t) = \dfrac{1}{a}(2a-t),\ a < t < 2a$.
   and $f(t) = f(t+2a)$. **(M.U. 2004)**
2. $f(t) = t,\ 0 < t < \pi$; $f(t) = \pi - t,\ \pi < t < 2\pi$
   and $f(t) = f(t+2\pi)$.
3. $f(t) = t^2,\ 0 < t < 2$, where $f(t)$ is a periodic function with period 2.
4. $f(t) = t,\ 0 < t < 1$ and $f(t)$ is of period 1.
5. $f(t) = \begin{cases} E, & 0 \le t \le (p/2) \\ -E, & (p/2) \le t \le p \end{cases}$, $f(t+p) = f(t)$. **(M.U. 2002, 03)**

[**Ans. :** (1) $\dfrac{1}{as^2} \tan h\left(\dfrac{as}{2}\right)$, (2) $\dfrac{1-(1+\pi s)e^{-\pi s}}{s^2(1+e^{-\pi s})}$,

(3) $\dfrac{2}{1-e^{-2s}}\left[\dfrac{1}{s^3} - \dfrac{1}{s^3} \cdot e^{-2s} - \dfrac{2}{s} \cdot e^{-2s} - \dfrac{2}{s^2} \cdot e^{-2s}\right]$,

(4) $\dfrac{1}{s^2} - \dfrac{e^{-s}}{s(1-e^{-s})}$, (5) $\dfrac{E}{s} \tan h\left(\dfrac{sp}{4}\right)$.]

5. Heaviside's Unit-step Function

The function defined by

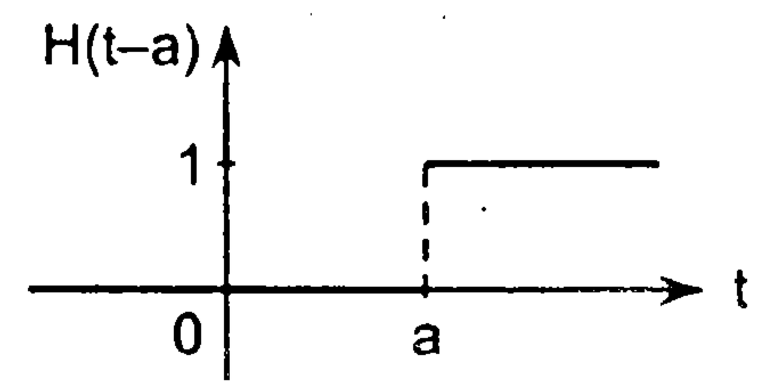
$$H(t-a) = 0 \text{ for } t < a$$

and $\quad H(t-a) = 1$ for $t \ge a$

is called **Heaviside's unit step function.** In particular if $a = 0$, we have the unit step function.

$$H(t) = 0 \text{ for } t < 0$$

and $\quad H(t) = 1$ for $t \ge 0$



Laplace transform of $H(t-a)$.

By definition of Laplace transform.

$$L\, H(t-a) = \int_0^{\infty} e^{-st} H(t-a)\, dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= \int_a^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{1}{s}e^{-as}$$

$$\therefore \quad \boxed{L\,H(t-a) = \frac{1}{s}e^{-as}} \text{ and } \boxed{L^{-1}\frac{1}{s}e^{-as} = H(t-a)} \quad \text{.......... (21)}$$

$$\text{If } a = 0, \quad \boxed{L\,H(t) = \frac{1}{s}} \text{ and } \boxed{L^{-1}\left(\frac{1}{s}\right) = H(t)} \quad \text{.......... (22)}$$

Laplace transform of $f(t-a)\cdot H(t-a)$. **(M.U. 1996, 2004)**

By definition of Laplace transform,

$$L\,f(t-a)\,H(t-a) = \int_0^\infty e^{-st} f(t-a)\,H(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)(0)\,dt + \int_a^\infty e^{-st} f(t-a)\cdot 1\cdot dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt = \int_0^\infty e^{-s(a+u)} f(u)\,du \qquad [t-a=u]$$

$$= e^{-as}\int_0^\infty e^{-su} f(u)\,du = e^{-as}\int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \quad \boxed{L[f(t-a)\,H(t-a)] = e^{-as}L\,f(t) = e^{-as}\,\Phi(s)} \quad \text{.......... (23)}$$

where, $\Phi(s) = L\,f(t)$

$$\text{And } \therefore \quad \boxed{L^{-1}\,e^{-as}\,\Phi(s) = f(t-a)\,H(t-a)} \quad \text{.......... (24)}$$

where, $f(t) = L^{-1}\,\Phi(s)$

$$\text{In particular if } a = 0, \quad \boxed{L[f(t)\,H(t)] = L\,f(t)} \quad \text{.......... (25)}$$

Ex. 1 : (i) Find Laplace transform of $t^2\,H(t-3)$.

(ii) Express $g(t) = (t-2)^2$ when $t > 2$ and $f(t) = 0$ when $0 < t < 2$ in terms of Heaviside unit step function and find its Laplace transform.

Sol. : (i) We first express $g(t) = t^2$ as a function of $(t-3)$ by using Taylor's series, which states that

$$f(x) = f(a) + (x-a)\cdot f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots.. + \ldots..$$

$$\therefore \quad g(t) = 9 + (t-3)\cdot 6 + \frac{(t-3)^3}{2!}\cdot 2$$

Now, by the above result,

$$L[g(t)\,H(t-a)] = L\left[9 + 6(t-3) + (t-3)^2\right]\cdot H(t-3)$$

$$= L\left[f(t-3)\cdot H(t-3)\right] = e^{-3s}L\,f(t) \quad \text{where, } f(t) = 9 + 6t + t^2.$$

$$\therefore \quad L[t^2\cdot H(t-a)] = e^{-3s}\left[\frac{9}{s} + \frac{6}{s^2} + \frac{2}{s^3}\right].$$

(ii) It is easy to see that

$$g(t) = (t-2)^2 = (t-2)^2\,H(t-2)$$

because for $t < 2$, $H(t-2) = 0$ and for $t \geq 2$, $H(t-2) = 1$.

$$\therefore \quad L[g(t)\,H(t-2)] = L\left[(t-2)^2\,H(t-2)\right]$$

$$= e^{-2s}\cdot L\,f(t) \quad \text{where, } f(t) = t^2$$

$$= e^{-2s}\cdot\frac{2!}{s^3}$$

Ex. 2 : Find the Laplace transform of

$$\sin t\cdot H\left(t-\frac{\pi}{2}\right) - H\left(t-\frac{3\pi}{2}\right). \qquad \textbf{(M.U. 1996)}$$

Sol. : Since, $L\,H(t-a) = \dfrac{1}{s}e^{-as}$ we have $L\,H\left(t-\dfrac{3\pi}{2}\right) = \dfrac{1}{s}e^{-3\pi s/2}$.

Now $\sin t = \sin\left(t - \dfrac{\pi}{2} + \dfrac{\pi}{2}\right) = \sin\left[\dfrac{\pi}{2} + \left(t-\dfrac{\pi}{2}\right)\right] = \cos\left(t-\dfrac{\pi}{2}\right)$

$$\therefore \quad L\sin t\cdot H\left(t-\frac{\pi}{2}\right) = L\left[\cos\left(t-\frac{\pi}{2}\right)\cdot H\left(t-\frac{\pi}{2}\right)\right]$$

Since, $L[f(t-a)\cdot H(t-a)] = e^{-as}\,L\,f(t)$

$$L\left[\cos\left(t-\frac{\pi}{2}\right)\cdot H\left(t-\frac{\pi}{2}\right)\right] = e^{-\pi s/2}\,L\cos t = e^{-\pi s/2}\cdot\frac{s}{s^2+1}$$

$$\therefore \quad L\left[\sin t\cdot H\left(t-\frac{\pi}{2}\right) - H\left(t-\frac{3\pi}{2}\right)\right] = e^{-\pi s/2}\cdot\frac{s}{s^2+1} - e^{-3\pi s/2}\cdot\frac{1}{s}.$$

Ex. 3 : Find Laplace transform of

$$(1 + 2t - 3t^2 + 4t^3)\,H(t-2). \qquad \textbf{(M.U. 1998)}$$

Sol. : We express $g(t) = 1 + 2t - 3t^2 + 4t^3$ as a function of $(t-2)$.

Now, $g(t) = 1 + 2t - 3t^2 + 4t^3$	$\therefore\ g(2) = 25$
$g'(t) = 2 - 6t + 12t^2$	$\therefore\ g'(2) = 38$
$g''(t) = -6 + 24t$	$\therefore\ g''(2) = 42$
$g'''(t) = 24$	$\therefore\ g'''(2) = 24$

$$\therefore \quad L[g(t)\,H(t-2)] = L\left[25 + 38(t-2) + \frac{42}{2!}(t-2)^2 + 24\frac{(t-2)^3}{3!}\right]H(t-2)$$

$$= L\,f(t)\cdot H(t-2)$$

where, $f(t) = 25 + 38t + 21t^2 + 4t^3$

$$\therefore \quad L[g(t)\,H(t-2)] = e^{-2s}\,L\left[25 + 38t + 21t^2 + 4t^3\right]$$

$$= e^{-2s}\left[\frac{25}{s} + \frac{38}{s^2} + \frac{21\cdot 2}{s^3} + \frac{4\cdot 3!}{s^4}\right]$$

$$= e^{-2s}\left[\frac{25}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4}\right]$$

Putting $g(t) = 1 + 2t - 3t^2 + 4t^3$

$$L(1 + 2t - 3t^2 + 4t^3)\,H(t-2) = e^{-2s}\left[\frac{25}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4}\right]$$

Ex. 4 : Using Laplace transform evaluate

$$\int_0^{\infty} e^{-t}(1 + 2t - 3t^2 + 4t^3)\,H(t-2).$$

Sol. : We have proved above that

$$L(1 + 2t - 3t^2 + 4t^3)\,H(t-2) = e^{-2s}\left[\frac{25}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4}\right]$$

By definition of Laplace transform, this means,

$$\int_0^{\infty} e^{-st}(1 + 2t - 3t^2 + 4t^3)\,H(t-2)\,dt = e^{-2s}\left[\frac{25}{s} + \frac{38}{s^2} + \frac{42}{s^3} + \frac{24}{s^4}\right]$$

Putting $s = 1$, we get

$$\int_0^{\infty} e^{-t}(1 + 2t - 3t^2 + 4t^3)\,H(t-2)\,dt$$

$$= e^{-2}\left[\frac{25}{1} + \frac{38}{1^2} + \frac{42}{1^3} + \frac{24}{1^4}\right] = \frac{129}{e^2}.$$

Ex. 5 : Using Laplace transform evaluate

$$\int_0^{\infty} e^{-t}(1 + 2t - t^2 + t^3)\,H(t-1)\,dt$$ **(M.U. 2003)**

Sol. : We shall first find the Laplace transform of $(1 + 2t - t^2 + t^3)\cdot H(t-1)$.

We express $g(t) = 1 + 2t - t^2 + t^3$ as a function of $(t-1)$.

Now, $g(t) = 1 + 2t - t^2 + t^3 \quad \therefore \; g(1) = 3$

$g'(t) = 2 - 2t + 3t^2 \quad \therefore \; g'(1) = 3$

$g''(t) = -2 + 6t \quad \therefore \; g''(1) = 4$

$g'''(t) = 6 \quad \therefore \; g'''(1) = 6$

$$\therefore \quad L[g(t)\cdot H(t-1)] = L\left[3 + 3(t-1) + \frac{4}{2!}(t-1)^2 + \frac{6}{3!}(t-1)^3\right]H(t-1)$$

$$= L[f(t)]\cdot H(t-1)$$

where, $f(t) = 3 + 3t + 2t^2 + t^3$.

$$\therefore \quad L[g(t)\cdot H(t-1)] = e^{-s}\cdot L[3 + 3t + 2t^2 + t^3]$$

$$= e^{-s}\left[\frac{3}{s} + \frac{3}{s^2} + 2\cdot\frac{2!}{s^3} + \frac{3!}{s^4}\right]$$

By definition of Laplace transform this means

$$\int_0^{\infty} e^{-st}(1 + 2t - t^2 + t^3)\cdot H(t-1)\,dt$$

$$= e^{-s}\left[\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right]$$

Putting $s = 1$,

$$\int_0^{\infty} e^{-t}(1 + 2t - t^2 + t^3)\cdot H(t-1)\,dt$$

$$= e^{-1}\left[\frac{3}{1} + \frac{3}{1^2} + \frac{4}{1^3} + \frac{6}{1^4}\right] = \frac{16}{e}$$

EXERCISE

1. Express the following functions in terms of Heaviside unit step function and hence, find the Laplace transform :

(i) $f(t) = (t-3)^4,\ t > 3,$
$\quad = 0,\ 0 < t < 3$

(ii) $f(t) = (t-4)^3,\ t > 4$
$\quad = 0,\ 0 < t < 4$

(iii) $f(t) = 1 + 2t - t^2,\ t > 2$
$\quad = 0,\ 0 < t < 2$

(iv) $f(t) = \cos t,\ 0 < t < a$
$\quad = \sin t,\ a < t < 2a$

2. Find the Laplace transform of the functions

(i) $t\,H(t-2)$, **(ii)** $\sin t\cdot H(t-\pi)$ **(iii)** $t^4\,H(t-2)$

(iv) $(1 + 3t - 4t^2 + 2t^3)\,H(t-3)$ **(v)** $t^2\,H(t-2)$ **(M.U. 1997)**

[**Ans. : (1) (i)** $(t-3)^4\,H(t-3)\;;\;\frac{24}{s^5}\cdot e^{-3s}$, **(ii)** $(t-4)^3\,H(t-4)\;;\;\frac{6}{s^4}\cdot e^{-4s}$,

(iii) $\left[1 - 2(t-2) - (t-2)^2\right]H(t-2)\;;\;e^{-2s}\left[\frac{1}{s} - \frac{2}{s^2} - \frac{2}{s^3}\right]$,

(iv) $\cos t\,H(t) + (\sin t - \cos t)\,H(t-\pi)\;;\;\frac{1}{s^2+1}\left[s + (s-1)\,e^{-\pi s}\right]$.

(2) (i) $e^{-2s}\left[\frac{2}{s}+\frac{1}{s^2}\right]$, (ii) $e^{-\pi s}\cdot\frac{-1}{s^2+1}$,

(iii) $e^{-4s}\left[\frac{16}{s}+\frac{32}{s^2}+\frac{48}{s^3}+\frac{48}{s^4}+\frac{24}{s^5}\right]$,

(iv) $e^{-3s}\left[\frac{28}{s}+\frac{33}{s^2}+\frac{28}{s^3}+\frac{12}{s^4}\right]$, (v) $e^{-2s}\left[\frac{4}{s}+\frac{4}{s^2}+\frac{2}{s^3}\right]$.]

Ex. 1 : Find inverse Laplace transform of

(i) $\frac{e^{-as}}{(s+b)^{5/2}}$ **(M.U. 2002),** (ii) $e^{-s}\left\{\frac{1-\sqrt{s}}{s^2}\right\}^2$

Sol. : (i) [By 24 page 4-56], $f(t)=L^{-1}\Phi(s)=L^{-1}\frac{1}{(s+b)^{5/2}}$

$$\therefore\quad f(t)=e^{-bt}\cdot L^{-1}\left(\frac{1}{s^{5/2}}\right)=e^{-bt}\cdot\frac{t^{3/2}}{\overline{|5/2}}=\frac{e^{-bt}\cdot t^{3/2}}{(3/2)\cdot(1/2)\overline{|1/2}}$$

$$=\frac{e^{-bt}\cdot t^{3/2}}{(3/2)\cdot(1/2)\sqrt{\pi}}=\frac{4}{3\sqrt{\pi}}\cdot e^{-bt}\cdot t^{3/2}$$

$$\therefore\quad L^{-1}\frac{e^{-as}}{(s+b)^{5/2}}=\frac{4}{3\sqrt{\pi}}\cdot e^{-b(t-a)}\cdot(t-a)^{3/2}\cdot H(t-a)$$

(ii) [By 24 page 4-56], $f(t)=L^{-1}\Phi(s)=L^{-1}\left(\frac{1-\sqrt{s}}{s^2}\right)^2$

$$\therefore\quad f(t)=L^{-1}\left[\frac{1-2\sqrt{s}+s}{s^4}\right]=L^{-1}\left[\frac{1}{s^4}-\frac{2}{s^{7/2}}+\frac{1}{s^3}\right]$$

$$=\frac{t^3}{3!}-2\cdot\frac{t^{5/2}}{\overline{|7/2}}+\frac{t^2}{2!}=\frac{t^3}{6}-\frac{16}{15\sqrt{\pi}}\cdot t^{5/2}+\frac{t^2}{2}\qquad(\because\ a=1)$$

$$\therefore\quad L^{-1}e^{-s}\left\{\frac{1-\sqrt{s}}{s}\right\}^2=\left[\frac{(t-1)^3}{6}-\frac{16}{15\sqrt{\pi}}(t-1)^{5/2}+\frac{(t-1)^2}{2}\right]H(t-1)$$

Ex. 2 : Find the inverse Laplace transform of the following.

(i) $\frac{e^{4-3s}}{(s+4)^{5/2}}$ **(M.U. 1997, 2000, 05),** (ii) $\frac{(s+1)e^{-s}}{s^2+s+1}$ **(M.U. 1997),**

(iii) $\frac{8e^{-3s}}{s^2+4}$ **(M.U. 1998),** (iv) $\frac{e^{-5s}}{(s-2)^4}$ **(M.U. 1999).**

Sol. : (i) [By 24 page 4-56], $f(t)=L^{-1}\Phi(s)=L^{-1}\frac{1}{(s+4)^{5/2}}$

$$=e^{-4t}L^{-1}\frac{1}{s^{5/2}}\qquad\text{(By first shifting theorem)}$$

$$=e^{-4t}\cdot\frac{t^{3/2}}{\overline{|5/2}}=\frac{e^{-4t}\cdot t^{3/2}}{(3/2)\cdot(1/2)\overline{|1/2}}$$

$$=\frac{4e^{-4t}\cdot t^{3/2}}{3\sqrt{\pi}}.$$

$$\therefore\quad L\frac{e^{4-3s}}{(s+4)^{5/2}}=\frac{e^4\,4}{3\sqrt{\pi}}e^{-4t}\cdot(t-3)^{3/2}H(t-3)$$

$$=\frac{4}{3\sqrt{\pi}}e^{-4(t-4)}\cdot(t-3)^{3/2}H(t-3)$$

(ii) [By 24 page 4-56],

$$f(t)=L^{-1}\Phi(s)=L^{-1}\left(\frac{s+1}{s^2+s+1}\right)=L^{-1}\frac{[s+(1/2)]+(1/2)}{[s+(1/2)]^2+(3/4)}$$

$$=L^{-1}\frac{[s+(1/2)]}{[s+(1/2)]^2+\left(\sqrt{3}/2\right)^2}+\frac{1}{2}L^{-1}\frac{1}{[s+(1/2)]^2+\left(\sqrt{3}/2\right)^2}$$

$$=e^{-t/2}L^{-1}\frac{s}{s^2+\left(\sqrt{3}/2\right)^2}+\frac{1}{2}e^{-t/2}L^{-1}\frac{1}{s^2+\left(\sqrt{3}/2\right)^2}$$

$$=e^{-t/2}\cos\left(\sqrt{3}t/2\right)+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}e^{-t/2}\sin\left(\sqrt{3}t/2\right)$$

$$\therefore\quad L^{-1}\frac{(s+1)e^{-s}}{s^2+s+1}$$

$$=e^{-(t-1)/2}\left[\cos\left(\sqrt{3}(t-1)/2\right)+\frac{1}{\sqrt{3}}\sin\left(\sqrt{3}(t-1)/2\right)\right]\cdot H(t-1)$$

(iii) [By 24 page 4-56],

$$f(t)=L^{-1}\Phi(s)=L^{-1}\frac{8}{s^2+4}=8\cdot\frac{1}{2}\sin 2t=4\sin 2t$$

$$\therefore\quad L\frac{8e^{-3s}}{s^2+4}=4\sin 2(t-3)\,H(t-3)$$

(iv) [By 24 page 4-56],

$$f(t)=L^{-1}\Phi(s)=L^{-1}\frac{1}{(s-2)^4}=e^{2t}L^{-1}\frac{1}{s^4}=e^{2t}\cdot\frac{t^3}{3!}$$

$$\therefore\quad L^{-1}\frac{e^{-5s}}{(s-2)^4}=\frac{1}{3!}e^{2(t-5)}\cdot(t-5)^3\cdot H(t-5).$$

Ex. 3 : Find the inverse Laplace transform of

(i) $\dfrac{e^{-\pi s}}{s^2-2s+2}$ **(M.U. 1995)** (ii) $\dfrac{e^{-3s}}{(s+4)^3}$ **(M.U. 1996)**

(iii) $\dfrac{e^{-2s}}{s^2+8s+25}$ **(M. U. 1999)**

Sol. : (i) $f(t) = L^{-1}\dfrac{1}{s^2-2s-2} = L^{-1}\dfrac{1}{(s-1)^2+1} = e^t\, L^{-1}\dfrac{1}{s^2+1} = e^t \sin t$

$$\therefore\ L^{-1}\frac{e^{-\pi s}}{s^2-2s+2} = e^{(t-\pi)}\cdot \sin(t-\pi)\cdot H(t-\pi)$$

(ii) $f(t) = L^{-1}\dfrac{1}{(s+4)^3} = e^{-4t}\, L^{-1}\dfrac{1}{s^3} = e^{-4t}\cdot\dfrac{t^2}{2}$

$$\therefore\ L^{-1}\frac{e^{-3s}}{(s+4)^3} = e^{-4(t-3)}\cdot\frac{(t-3)^2}{2}\cdot H(t-3)$$

(iii) $f(t) = L^{-1}\dfrac{1}{(s^2+8s+25)} = L^{-1}\dfrac{1}{(s+4)^2+3^2}$

$$= e^{-4t}\, L^{-1}\frac{1}{s^2+3^2} = e^{-4t}\cdot\frac{1}{3}\sin 3t$$

$$\therefore\ L^{-1}\left(\frac{e^{-2s}}{s^2+8s+2s}\right) = \frac{1}{3}e^{-4(t-2)}\sin 3(t-2)\cdot H(t-2).$$

EXERCISE

Find the inverse transform of

1. $\dfrac{e^{-s}}{(s+1)^2}$, 2. $\dfrac{e^{-\pi s}}{(s^2+9)}$, 3. $\dfrac{s\,e^{-as}}{s^2+3s+2}$, 4. $e^{-s}\dfrac{(1+\sqrt{s})}{s^3}$,

5. $\dfrac{e^{-as}}{s-b}$ 6. $\dfrac{s\,e^{-as}}{s^2+b^2}$, 7. $\dfrac{s\,e^{-s/2}+\pi\, e^{-s}}{s^2+\pi^2}$, 8. $\dfrac{s\,e^{-3s}}{s^2-1}$,

9. $\dfrac{e^{-bs}}{s^2(s+a)}$, 10. $\dfrac{e^{-\pi s}}{s^2(s^2+1)}$,

11. $\dfrac{e^{-4s}}{\sqrt{2s+7}}$ **(M.U. 2003)** 12. $\dfrac{s\cdot e^{-2s}}{s^2+2s+2}$ **(M.U. 2003)**

[**Ans. : (1)** $e^{-(t-1)}(t-1)H(t-1)$, **(2)** $\dfrac{1}{3}\sin 3(t-\pi)\,H(t-\pi)$,

(3) $\left[2e^{-2(t-a)} - e^{-(t-a)}\right]H(t-a)$, **(4)** $\left[\dfrac{(t-1)^2}{2} + \dfrac{4(t-1)^{3/2}}{3\sqrt{\pi}}\right]H(t-1)$,

(5) $e^{b(t-a)}\cdot H(t-a)$, **(6)** $\cos b(t-a)\cdot H(t-a)$,

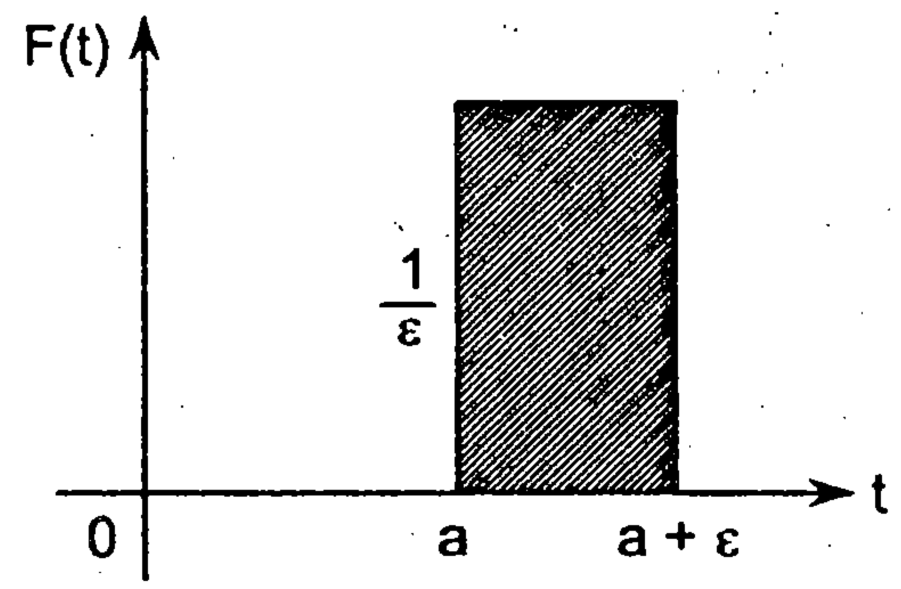
(7) $\sin\pi t\left[H\left(t-\dfrac{1}{2}\right)+H(t-1)\right]$, **(8)** $\cos h(t-3)\cdot H(t-3)$,

(9) $\dfrac{1}{a^2}\left[a(t-b)-1+e^{-a(t-b)}\right]\cdot H(t-b)$,

(10) $[(t-\pi)+\sin(t-\pi)\cdot H(t-\pi)]$, **(11)** $\dfrac{e^{-7(t-4)/2}}{\sqrt{2\pi(t-4)}}\cdot H(t-4)$

(12) $e^{-(t-2)}[\cos(t-2)-\sin(t-2)]H(t-2)$]

6. Dirac-delta Function (Unit-impulse Function)



Consider the function $F(t)$ defined by,

$$F(t) = 0,\ t < a$$
$$= \frac{1}{\varepsilon},\ a \le t \le a+\varepsilon$$
$$= 0,\ t > a+\varepsilon$$

The function is represented by the adjoining figure.

Integrating $F(t)$, we get,

$$\int_0^\infty F(t)\,dt = \int_a^{a+\varepsilon}\frac{1}{\varepsilon}\,dt = 1 \quad \text{for all } \varepsilon.$$

As $\varepsilon \to 0$, the function $F(t)$ tends to infinity at a and is zero everywhere else. But the integral of $F(t)$ is unity.

If $F(t)$ represents a force acting for a short time ε at time $t = a$ then the integral $\lim\limits_{\varepsilon\to 0}\int_0^\infty F(t)\,dt\,(=1)$ represents **unit-impulse** at $t = a$. Hence, the limiting form of $F(t)$ (as $\varepsilon \to 0$) is known as **unit impulse function or Dirac-delta function** and is denoted by $\delta(t-a)$.

$$\therefore\ \delta(t-a) = \lim_{\varepsilon\to 0} F(t)$$

When $a = 0$, the unit function at $t = 0$ is $\boxed{\delta(t) = \lim\limits_{\varepsilon\to 0} F(t)}$

6.1 Laplace transform of Dirac-delta function $L[\delta(t-a)]$

By definition of Laplace transform,

$$L[F(t)] = \int_0^\infty e^{-st}F(t)\,dt = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} e^{-st}dt$$

$$= \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon} = -\frac{1}{\varepsilon s}\left[e^{-s(a+\varepsilon)} - e^{-as}\right]$$

$$= \frac{1}{s} \cdot e^{-as}\left[\frac{1-e^{-\varepsilon s}}{\varepsilon}\right]$$

$$\therefore \quad L[\delta(t-a)] = \lim_{\varepsilon\to 0} L[F(t)] = \frac{1}{s} e^{-as} \lim_{\varepsilon \to 0}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon}\right)$$

$\therefore$ $\boxed{L[\delta(t-a)] = e^{-as}}$ (By L'Hospital's Rule) .... (26)

Cor. : Putting $a = 0$, $\boxed{L[\delta(t)] = 1}$ .... (27)

Inverse Laplace transform : From the above results, we get,

$$\boxed{L^{-1}(e^{-as}) = \delta(t-a) \text{ and } L^{-1}(1) = \delta(t)} \quad \text{.......... (28)}$$

6.2 Laplace transform of $f(t)\,\delta(t-a)$

By definition, $L f(t)\,\delta(t-a) = \lim_{\varepsilon\to 0} \int_0^\infty e^{-st} f(t) \cdot F(t)\, dt$

Now, $\int_0^\infty e^{-st} f(t) \cdot F(t)\, dt = \frac{1}{\varepsilon} \int_a^{a+\varepsilon} e^{-st} f(t) \cdot 1 \cdot dt$

$$= \frac{1}{\varepsilon} \cdot \varepsilon \cdot e^{-s(a+\theta\varepsilon)} \cdot f(a+\theta\varepsilon) \quad \text{.......... (1)}$$

[By mean value theorem for integrals,

$$\int_0^{a+h} f(x)\, dx = h\, f(a+\theta h), \quad 0 < \theta < 1.]$$

Taking the limit as $\varepsilon \to 0$, from (1), we get,

$$\boxed{L f(t)\,\delta(t-a) = e^{-as} f(a)} \quad \text{.......... (29)}$$

Ex. 1 : Find Laplace transform of $\sin 2t \cdot \delta(t-2)$.

Sol. : Taking $f(t) = \sin at$ and $a = 2$.

$$L \sin 2t \cdot \delta(t-2) = L f(t)\,\delta(t-2) = e^{-as} f(a)$$

$$= e^{-2s} \sin 4.$$

Ex. 2 : Find Laplace transform of $t[H(t-4)] + t^2\,\delta(t-4)$.

Sol. : For, $L[t\,H(t-4)]$ we write $t = 4 + (t-4)$

$$\therefore \quad L[t\,H(t-4)] = L[\{4+(t-4)\}\,H(t-4)]$$

$$= L[f(t-4)\,H(t-4)]$$

$$= e^{-4s} L f(t) \quad \text{where, } f(t) = 4 + t.$$

$$\therefore \quad L[t\,H(t-4)] = e^{-4s}\left[\frac{4}{s} + \frac{1}{s^2}\right]$$

For, $L[t^2\,\delta(t-4)]$, , we take $f(t) = t^2$ and $a = 4$.

$$\therefore \quad L[t^2\,\delta(t-4)] = L f(t)\,\delta(t-4) = e^{-as} f(a)$$

$$= e^{-4s} \cdot 16.$$

$$\therefore \quad L\left[t \cdot H(t-4) + t^2\,\delta(t-4)\right] = e^{-4s}\left[\frac{4}{s} + \frac{1}{s^2}\right] + 16 e^{-4s}$$

$$= \frac{e^{-4s}}{s^2}\left[1 + 4s + 16s^2\right]$$

EXERCISE

Find Laplace transform of

1. $\sin t \cdot \delta\left(t - \frac{\pi}{2}\right) - t^2\,\delta(t-2)$. 2. $t^4\,H(t-2) + t^2\,\delta(t-2)$.

[**Ans. :** (1) $e^{-\pi s/2} - 4e^{-2s}$, (2) $e^{-2s}\left[4 + \frac{16}{s} + \frac{32}{s^2} + \frac{48}{s^3} + \frac{48}{s^4} + \frac{42}{s^5}\right]$.]

7. Applications Of Laplace Transforms

In this article we shall see how Laplace transforms can be profitably used to solve differential equations. For the use of Laplace transforms to solve differential equations we need the following results. If Laplace transform of y *i.e.* $L(y)$ is denoted by $\bar{y}$ then we have

$$\boxed{\begin{aligned} L(y') &= s\bar{y} - y(0) \\ L(y'') &= s^2\bar{y} - sy(0) - y'(0) \\ L(y''') &= s^3\bar{y} - s^2 y - sy'(0) - y''(0) \end{aligned}} \quad \text{.......... (30)}$$

Ex. 1 : Solve $3\frac{dy}{dt} + 2y = e^{3t}$, $y = 1$ at $t = 0$.

Sol. : Taking Laplace transforms of both sides,

$$3L(y') + 2L(y) = L(e^{3t})$$

$$3[s\bar{y} - y(0)] + 2\bar{y} = \frac{1}{s-3}. \quad \text{But } y(0) = 1$$

$$\therefore \quad 3[s\bar{y} - 1] + 2\bar{y} = \frac{1}{s-3}$$

$$\therefore \quad (3s+2)\,\bar{y} = \frac{1}{s-3} + 3 = \frac{3s-8}{s-3}$$

$$\bar{y} = \frac{3s-8}{(s-3)(3s+2)}$$

$$\therefore \quad \bar{y} = \frac{30}{11} \cdot \frac{1}{3s+2} + \frac{1}{11} \cdot \frac{1}{(s-3)}$$

(By partial fractions),

$$\therefore \quad \bar{y} = \frac{10}{11} \cdot \frac{1}{s+(2/3)} + \frac{1}{11} \cdot \frac{1}{s-3}$$

Taking inverse Laplace transforms

$$y = \frac{10}{11} L^{-1}\left[\frac{1}{s+(2/3)}\right] + \frac{1}{11} L^{-1}\left[\frac{1}{s-3}\right]$$

$$= \frac{10}{11} e^{-(2/3)t} + \frac{1}{11} e^{3t}.$$

Ex. 2 : Solve $\frac{dy}{dx} + 3y = 2 + e^{-t}$, if $y = 1$ at $t = 0$.

Sol. : Taking Laplace transforms of both sides,

$$L(y') + 3L(y) = L(2) + L(e^{-t})$$

$$s\bar{y} - y(0) + 3\bar{y} = 2\frac{1}{s} + \frac{1}{s+1}. \quad \text{But } y(0) = 1$$

$$\therefore \quad (s+3)\bar{y} = \frac{2}{s} + \frac{1}{s+1} + 1 = \frac{s^2+4s+2}{s(s+1)}$$

$$\therefore \quad \bar{y} = \frac{s^2+4s+2}{s(s+1)(s+3)}$$

$\therefore$ By partial fractions

$$\bar{y} = \frac{2}{3} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s+1} - \frac{1}{6} \cdot \frac{1}{s+3}$$

Taking inverse Laplace transforms,

$$y = \frac{2}{3} + \frac{1}{2} \cdot e^{-t} - \frac{1}{6} e^{-3t}$$

Ex. 3 : Solve $R\frac{dQ}{dt} + \frac{Q}{C} = V$, $Q = 0$ when $t = 0$.

Sol. : Taking Laplace transforms of both sides,

$$RL(Q') + \frac{1}{C} L(Q) = VL(1)$$

$$R\left[s\bar{Q} - Q(0)\right] + \frac{1}{C}\bar{Q} = \frac{V}{s}. \quad \text{But } Q(0) = 0$$

$$\therefore \quad \left(Rs + \frac{1}{C}\right)\bar{Q} = \frac{V}{s} \qquad \therefore \quad \bar{Q} = \frac{V}{s} \cdot \frac{1}{[Rs + (1/C)]} = \frac{VC}{s(RCs+1)}$$

$$\therefore \quad \bar{Q} = VC\,L^{-1}\frac{1}{s} - VRC^2\,L^{-1}\frac{1}{RCs+1}$$

$$= VC\,L^{-1}\frac{1}{s} - VC\,L^{-1}\left[\frac{1}{s+1/(RC)}\right]$$

Taking inverse Laplace transforms,

$$Q = VC - VC\,e^{-t/RC}$$

$$= VC(1 - e^{-t/RC})$$

Ex. 4 : Solve $L\frac{dI}{dt} + RI = E\,e^{-at}$, where $I(0) = 0$.

Sol. : Taking Laplace transforms of both sides,

$$L\,L[I'] + RL(I) = EL(e^{-at})$$

$$L[s\bar{I} - I(0)] + R\bar{I} = \frac{E}{s+a}. \quad \text{But } I(0) = 0$$

$$\therefore \quad (Ls+R)\bar{I} = \frac{E}{s+a}$$

$$\therefore \quad \bar{I} = \frac{E}{(s+a)(Ls+R)} \qquad \text{(By partial fractions)}$$

$$= \frac{E}{R-La} \cdot \frac{1}{s+a} - \frac{EL}{R-La} \cdot \frac{1}{Ls+R}$$

Taking inverse Laplace transforms

$$I = \frac{E}{R-La} e^{-at} - \frac{E}{R-La} e^{-(R/L)t}$$

$$= \frac{E}{R-La}\left[e^{-at} - e^{-(R/L)t}\right]$$

Ex. 5 : Using Laplace transform solve the following differential equation.

$$\frac{dx}{dt} + x = \sin\omega t, \; x(0) = 2. \qquad \textbf{(M.U. 1993)}$$

Sol. : Let $\bar{x}$ be the Laplace transform of x *i.e.* let $L(x) = \bar{x}$.

Taking Laplace transform of both sides of the given equation,

$$L(x') + L(x) = L\sin(\omega t) \qquad \text{.......... (1)}$$

But $\quad L(x') = s(\bar{x}) - x(0) = s\bar{x} - 2$

Hence, the equation (1) becomes,

$$s\bar{x} - 2 + \bar{x} = \frac{\omega}{s^2+\omega^2} \qquad \therefore \quad (s+1)\bar{x} = 2 + \frac{\omega}{s^2+\omega^2}$$

$$\therefore \quad (s+1)\,\bar{x} = \frac{2s^2 + 2\omega^2 + \omega}{s^2 + \omega^2}$$

$$\therefore \quad \bar{x} = \frac{2s^2 + 2\omega^2 + \omega}{(s^2 + \omega^2)(s+1)} \quad \text{(By partial fractions)}$$

$$= \frac{2\omega^2 + \omega + 2}{1 + \omega^2} \cdot \frac{1}{s+1} + \frac{-\omega s + \omega}{(1+\omega^2)(s^2 + \omega^2)}$$

$$= \frac{2\omega^2 + \omega + 2}{1 + \omega^2} \cdot \frac{1}{s+1} - \frac{\omega}{1+\omega^2} \cdot \frac{s}{s^2 + \omega^2} + \frac{\omega}{1+\omega^2} \cdot \frac{1}{s^2 + \omega^2}$$

Taking inverse Laplace transforms,

$$x = \frac{2\omega^2 + \omega + 2}{1+\omega^2} L^{-1}\left(\frac{1}{s+1}\right) - \frac{\omega}{1+\omega^2} \cdot L^{-1}\left(\frac{s}{s^2 + \omega^2}\right) + \frac{\omega}{1+\omega^2} L^{-1}\left(\frac{1}{s^2+\omega^2}\right)$$

$$= \frac{2\omega^2 + \omega + 2}{1+\omega^2} \cdot e^{-t} - \frac{\omega}{1+\omega^2} \cdot \cos \omega t + \frac{\omega}{1+\omega^2} \cdot \frac{1}{\omega} \sin \omega t$$

$$= \frac{1}{1+\omega^2}\left[(2\omega^2 + \omega + 2)\, e^{-t} - \omega \cos \omega t + \sin \omega t\right]$$

Ex. 6 : Solve $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$ where $x = 0$, $y = 2$ at $t = 0$.

(M.U. 1993)

Sol. : Taking Laplace transforms of both the equations, we get,

$$L(x') + L(y) = L(\sin t)$$

and $L(y') + L(x) = L(\cos t)$

If $L(x) = \bar{x}$ and $L(y) = \bar{y}$ then we have

$$L(x') = s(\bar{x}) - x(0) = s\,\bar{x}$$

and $L(y') = s(\bar{y}) - y(0) = s\bar{y} - 2$

$$\therefore \quad s\bar{x} + \bar{y} = \frac{1}{s^2+1} \qquad \text{..........(1)}$$

$$s\bar{y} - 2 + \bar{x} = \frac{s}{s^2+1} \quad \therefore \quad s\bar{y} + \bar{x} = \frac{s}{s^2+1} + 2$$

$$\therefore \quad s\bar{y} + \bar{x} = \frac{2s^2 + s + 2}{s^2+1} \qquad \text{..........(2)}$$

Multiply (1) by s and subtract it from (2)

$$\therefore \quad s^2\,\bar{x} + s\,\bar{y} = \frac{s}{s^2+1}$$

$$s\bar{y} + \bar{x} = \frac{2s^2 + s + 2}{s^2+1}$$

$$\therefore \quad s^2\,\bar{x} - \bar{x} = \frac{-2s^2 - 2}{(s^2+1)} \quad \therefore \quad \bar{x}(s^2 - 1) = -2\frac{(s^2+1)}{(s^2+1)} = -2$$

$$\therefore \quad \bar{x} = -\frac{2}{s^2 - 1} \qquad \text{..........(3)}$$

$$\therefore \quad x = L^{-1}\frac{-2}{(s^2-1)} = -2 \sin ht.$$

Putting the value of $\bar{x}$ from (3) in (1)

$$-\frac{2s}{s^2-1} + \bar{y} = \frac{1}{s^2+1} \quad \therefore \quad \bar{y} = \frac{1}{s^2+1} + \frac{2s}{s^2-1}$$

$$\therefore \quad y = L^{-1}\left(\frac{1}{s^2+1}\right) + 2\,L^{-1}\frac{s}{s^2-1}$$

$$= \sin t + 2 \cos ht.$$

Ex. 7 : Solve $Dx - 2y - x = -2te^{-t} + e^{t} - 6t$,

$D^2x - Dy = te^{-t} - 2e^{-t} - 3$ given that $x = 0$, $Dx = 1$, $y = 0$ when $t = 0$.

Sol. : Taking Laplace transforms of both the equations we get,

$$L(x') - 2L(y) - L(x) = -2L(te^{-t}) + L(e^{-t}) - 6L(t)$$

$$L(x'') - L(y') = L(te^{-t}) - 2L(e^{-t}) - 3L(1)$$

If $L(x) = \bar{x}$ and $L(y) = \bar{y}$, we have

$$L(x') = s\,\bar{x} - x(0) = s\,\bar{x}$$

$$L(x'') = s^2\bar{x} - s\,x(0) - x'(0) = s^2\bar{x} - 1$$

$$L(y') = s\,\bar{y} - y(0) = s\,\bar{y}$$

$$L(t\,e^{-t}) = (-1)\frac{d}{ds}\left[L(e^{-t})\right] = -\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{1}{(s+1)^2}$$

$$L(e^{-t}) = \frac{1}{s+1}, \quad L(t) = \frac{1}{s^2}, \quad L(1) = \frac{1}{s}$$

$\therefore$ The equations become

$$s\bar{x} - 2\bar{y} - \bar{x} = -\frac{2}{(s+1)^2} + \frac{1}{s+1} - \frac{6}{s^2}$$

$$i.e.\quad (s-1)\,\bar{x}-2\bar{y}=\frac{s-1}{(s+1)^2}-\frac{6}{s^2} \qquad \text{.......... (1)}$$

$$\text{And}\quad s^2\bar{x}-1-s\bar{y}=\frac{1}{(s+1)^2}-\frac{2}{s+1}-\frac{3}{s}$$

$$s^2\bar{x}-s\bar{y}=1+\frac{1}{(s+1)^2}-\frac{2}{s+1}-\frac{3}{s}$$

$$=\frac{s^2+2s+1+1-2s-2}{(s+1)^2}-\frac{3}{s}$$

$$i.e.\quad s^2\bar{x}-s\bar{y}=\frac{s^2}{(s+1)^2}-\frac{3}{s} \qquad \text{.......... (2)}$$

Now, multiply (1) by s and (2) by 2 and subtract,

$$\therefore\ (s^2+s)\,\bar{x}=\frac{2s^2}{(s+1)^2}-\frac{6}{s}-\frac{s(s-1)}{(s+1)^2}+\frac{6}{s}$$

$$\therefore\ s(s+1)\,\bar{x}=\frac{s(s+1)}{(s+1)^2} \quad \therefore\ \bar{x}=\frac{1}{(s+1)^2}$$

$$\therefore\ x=L^{-1}\frac{1}{(s+1)^2}=e^{-t}L\frac{1}{s^2}=t\,e^{-t}$$

Now, putting the value of $\bar{x}$ in (2) we get,

$$s^2\cdot\frac{1}{(s+1)^2}-s\bar{y}=\frac{s^2}{(s+1)^2}-\frac{3}{s}$$

$$\therefore\ s\,\bar{y}=\frac{3}{s} \quad \therefore\ \bar{y}=\frac{3}{s^2}$$

$$\therefore\ y=L^{-1}\frac{3}{s^2}=3t.$$

Ex. 8 : Solve the following equations

$$D^2x+3x-2y=0,\ D^2x+D^2y-3x+5y=0$$

and $Dx=3,\ Dy=2,\ x=0,\ y=0$ at $t=0$.

Sol. : Taking Laplace transforms of both sides

$$L(x'')+3L(x)-2L(y)=0$$

$$L(x'')+L(y'')-3L(x)+5L(y)=0$$

If $L(x)=\bar{x}$ and $L(y)=\bar{y}$, we get

$$L(x')=s(\bar{x})-x(0)=s\,\bar{x}$$

$$L(x'')=s^2\bar{x}-s\,x(0)-x'(0)=s^2\bar{x}-3$$

$$L(y')=s\,\bar{y}-y(0)=s\,\bar{y}$$

$$L(y'')=s^2\bar{y}-s\,y(0)-y'(0)=s^2\bar{y}-2$$

$\therefore$ The equations become

$$s^2\bar{x}-3+3\bar{x}-2\bar{y}=0$$

$$\text{and}\quad s^2\bar{x}-3+s^2\,\bar{y}-2-3\bar{x}+5\bar{y}=0$$

$$\therefore\ (s^2+3)\,\bar{x}-2\bar{y}=3 \qquad \text{.......... (1)}$$

$$\text{and}\quad (s^2-3)\,\bar{x}+(s^2+5)\,\bar{y}=5 \qquad \text{.......... (2)}$$

Multiply (1) by s^2+5 and (2) by 2 and add

$$(s^2+5)(s^2+3)\,\bar{x}-2(s^2+5)\,\bar{y}=3(s^2+5)$$

$$2(s^2-3)\,\bar{x}+2(s^2+5)\,\bar{y}=10$$

$$\therefore\ \left[(s^2+5)(s^2+3)+2(s^2-3)\right]\bar{x}=3s^2+25$$

$$\therefore\ \left[s^4+8s^2+15+2s^2-6\right]\bar{x}=3s^2+25$$

$$\therefore\ \left[s^4+10s^2+9\right]\bar{x}=3s^2+25$$

$$\therefore\ (s^2+9)(s^2+1)\,\bar{x}=3s^2+25$$

$$\therefore\ \bar{x}=\frac{3s^2+25}{(s^2+9)(s^2+1)}$$

Let $s^2=u$ and $\dfrac{3u+25}{(u+9)(u+1)}=\dfrac{a}{u+9}+\dfrac{b}{u+1}$

$$\therefore\ 3u+25=a(u+1)+b(u+9)$$

When $u=-1,\ 22=8b \quad \therefore\ b=11/4$

When $u=-9,\ -2=-8a \quad \therefore\ a=1/4$

$$\therefore\ \bar{x}=\frac{1}{4}\cdot\frac{1}{s^2+9}+\frac{11}{4}\cdot\frac{1}{s^2+1}$$

Taking inverse Laplace transform

$$x=\frac{1}{4}L^{-1}\frac{1}{s^2+9}+\frac{11}{4}L^{-1}\frac{1}{s^2+1}$$

$$=\frac{1}{12}\sin 3t+\frac{11}{4}\sin t$$

Now multiply (1) by (s^2-3) and (ii) by (s^2+3) and subtract

$$\therefore\ (s^2-3)(s^2+3)\,\bar{x}-2(s^2-3)\,\bar{y}=3(s^2-3)$$

$$(s^2+3)(s^2-3)\,\bar{x}+(s^2+5)(s^2+3)\,\bar{y}=5(s^2+3)$$

$$\therefore\ \left[(s^2+5)(s^2+3)+2(s^2-3)\right]\bar{y}=2s^2+24$$

$$\therefore \left[s^4 + 8s^2 + 15 + 2s^2 - 6\right]\bar{y} = 2(s^2 + 12)$$

$$(s^4 + 10s^2 + 9)\,\bar{y} = 2(s^2 + 12)$$

$$\therefore \bar{y} = \frac{2(s^2 + 12)}{(s^2 + 9)(s^2 + 1)}$$

Let $s^2 = u$ and $\dfrac{u + 12}{(u + 9)(u + 1)} = \dfrac{a}{u + 9} + \dfrac{b}{u + 1}$

$$\therefore u + 12 = a(u + 1) + b(u + 9)$$

When $u = -1$, $11 = 8b$ $\quad\therefore b = 11/8$

When $u = -9$, $3 = -8a$ $\quad\therefore a = -3/8$

$$\therefore \bar{y} = -\frac{3}{4}\cdot\frac{1}{s^2 + 9} + \frac{11}{4}\cdot\frac{1}{s^2 + 1}$$

Taking inverse Laplace transform

$$y = -\frac{3}{4}L^{-1}\frac{1}{s^2 + 9} + \frac{11}{4}L^{-1}\frac{1}{s^2 + 1}.$$

$$\therefore y = -\frac{1}{4}\sin 3t + \frac{11}{4}\sin t.$$

Ex. 9 : Solve $(D^2 - 3D + 2)\,y = 4e^{2t}$, with $y(0) = -3$ and $y'(0) = 5$.

(M.U. 2004)

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') - 3L(y') + 2L(y) = 4L(e^{2t})$$

But $\quad L(y') = s\bar{y} - y(0) = s\bar{y} + 3$

and $\quad L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} + 3s - 5$

$\therefore$ The equation becomes,

$$(s^2\bar{y} + 3s - 5) - 3(s\bar{y} + 3) + 2\bar{y} = 4\frac{1}{s - 2}$$

$$(s^2 - 3s + 2)\,\bar{y} = \frac{4}{s - 2} + 14 - 3s = \frac{-3s^2 + 20s - 24}{s - 2}$$

$$\therefore \bar{y} = \frac{-3s^2 + 20s - 24}{(s - 2)(s^2 - 3s + 2)} = \frac{-3s^2 + 20s - 24}{(s - 1)(s - 2)^2}$$

By partial fractions, $\quad \bar{y} = -\dfrac{7}{s - 1} + \dfrac{4}{s - 2} + \dfrac{4}{(s - 2)^2}$

Taking inverse Laplace transform,

$$y = -7L^{-1}\left(\frac{1}{s - 1}\right) + 4L^{-1}\frac{1}{s - 2} + 4L^{-1}\frac{1}{(s - 2)^2}$$

$$= -7e^t L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s^2} = -7e^t + 4e^{2t} + 4te^{2t}$$

$\therefore$ The solution is $y = -7e^t + 4e^{2t} + 4te^{2t}$.

Ex. 10 : Solve $(D^2 - D - 2)\,y = 20\sin 2t$, with $y(0) = 1$ and $y'(0) = 2$.

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') - L(y') - 2L(y) = 20L(\sin 2t)$$

But $\quad L(y') = s\bar{y} - y(0) = s\bar{y} - 1$

and $\quad L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - s - 2$

$\therefore$ The equation becomes,

$$(s^2\bar{y} - s - 2) - (s\bar{y} - 1) - 2\bar{y} = 20\frac{2}{s^2 + 4}$$

$$\therefore (s^2 - s - 2)\,\bar{y} = \frac{40}{s^2 + 4} + s + 1 = \frac{s^3 + s^2 + 4s + 44}{s^2 + 4}$$

$$\therefore \bar{y} = \frac{s^3 + s^2 + 4s + 44}{(s^2 + 4)(s^2 - s - 2)}$$

$$= -\frac{8}{3}\cdot\frac{1}{s + 1} + \frac{8}{3}\cdot\frac{1}{s - 2} + \frac{s - 6}{s^2 + 4}$$

Taking inverse Laplace transform,

$$y = \frac{-8}{3}L^{-1}\left(\frac{1}{s + 1}\right) + \frac{8}{3}L^{-1}\left(\frac{1}{s - 2}\right) + L^{-1}\frac{s}{s^2 + 4} - 6L^{-1}\frac{1}{s^2 + 4}$$

$$\therefore y = -\frac{8}{3}e^{-t} + \frac{8}{3}e^{-2t} + \cos 2t - 3\sin 2t.$$

Ex. 11 : Solve $(D^2 + 3D + 2)\,y = 2(t^2 + t + 1)$ with $y(0) = 2$ and $y'(0) = 0$.

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') + 3L(y') + 2L(y) = 2L(t^2 + t + 1)$$

But $\quad L(y') = s\bar{y} - y(0) = s\bar{y} - 2$

and $\quad L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 2s$

$\therefore$ The equation becomes,

$$(s^2\bar{y} - 2s) + 3(s\bar{y} - 2) + 2\bar{y} = 2\left(\frac{2}{s^3} + \frac{1}{s^2} + \frac{1}{s}\right)$$

$$\therefore (s^2 + 3s + 2)\,\bar{y} = 2\frac{(2 + s + s^2)}{s^3} + 2s + 6$$

$$= \frac{2(s^4 + 3s^3 + s^2 + s + 2)}{s^3}$$

By partial fractions,

$$\therefore \quad \bar{y} = \frac{2(s^4 + 3s^3 + s^2 + s + 2)}{s^3(s^2 + 3s + 2)} = \frac{3}{s} - \frac{2}{s^2} + \frac{2}{s^3} - \frac{1}{s+2}$$

Taking inverse Laplace transform,

$$y = 3L^{-1}\frac{1}{s} - 2L^{-1}\frac{1}{s^2} + 2L^{-1}\frac{1}{s^3} - L^{-1}\frac{1}{s+2}$$

$$y = 3 - 2t + t^2 - e^{-2t}$$

Ex. 12 : Use Laplace transform to solve,

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = 1 \text{ where, } y(0) = 0,\ y'(0) = 1. \qquad \textbf{(M.U. 1994)}$$

Sol. : Let $\bar{y}$ be the Laplace transform of y *i.e.* let $L(y) = \bar{y}$.

Taking Laplace transform of the both sides,

$$L(y'') + 4L(y') + 8L(y) = L(1) \qquad \text{..........}(1)$$

Now; $L(y') = s\bar{y} - y(0) = s\bar{y}$

$$L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1 \text{ and } L(1) = \frac{1}{s}$$

$\therefore$ The equation (1) becomes

$$s^2\bar{y} - 1 + 4s\bar{y} + 8\bar{y} = \frac{1}{s}$$

$$\therefore \quad \bar{y}(s^2 + 4s + 8) = \frac{1}{s} + 1 = \frac{s+1}{s}$$

$$\therefore \quad \bar{y} = \frac{s+1}{s(s^2 + 4s + 8)} \qquad \therefore \quad y = L^{-1}(\bar{y}) = L^{-1}\frac{s+1}{s(s^2 + 4s + 8)}$$

We obtain the L^{-1} by partial fractions

$$\therefore \quad y = L^{-1}\left[\frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s}{s^2 + 4s + 8} + \frac{1}{2}\cdot\frac{1}{s^2 + 4s + 8}\right]$$

$$= \frac{1}{8}L^{-1}\left(\frac{1}{s}\right) - \frac{1}{8}L^{-1}\frac{(s+2) - 2}{(s+2)^2 + 2^2} + \frac{1}{2}L^{-1}\frac{1}{(s+2)^2 + 2^2}$$

$$= \frac{1}{8}\cdot 1 - \frac{1}{8}e^{-2t}L^{-1}\frac{s}{s^2 + 2^2} + \frac{6}{8}e^{-2t}L^{-1}\frac{1}{s^2 + 2^2}$$

$$\therefore \quad y = \frac{1}{8} - \frac{1}{8}e^{-2t}\cos 2t + \frac{3}{8}e^{-2t}\sin 2t.$$

Ex. 13 : Using Laplace transform solve

$$\frac{d^2y}{dt^2} + y = t,\ y(0) = 1,\ y'(0) = 0. \qquad \textbf{(M.U. 1995)}$$

Sol. : Let $\bar{y}$ be the Laplace transform of y *i.e.* let $L(y) = \bar{y}$.

Taking Laplace transform of both sides,

$$L(y'') + L(y) = L(t)$$

Now, $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - s$ and $L(t) = \frac{1}{s^2}$.

$\therefore$ The equation (1) becomes

$$s^2\bar{y} - s + \bar{y} = \frac{1}{s^2} \qquad \therefore \quad s^2\bar{y} + \bar{y} = s + \frac{1}{s^2} = \frac{s^3 + 1}{s^2}$$

$$\therefore \quad (s^2 + 1)\bar{y} = \frac{s^3 + 1}{s^2} \qquad \therefore \quad \bar{y} = \frac{s^3 + 1}{s^2(s^2 + 1)}$$

Let $\quad \dfrac{s^3 + 1}{s^2(s^2 + 1)} = \dfrac{a}{s} + \dfrac{b}{s^2} + \dfrac{cs + d}{s^2 + 1}$

$$\therefore \quad s^3 + 1 = as(s^2 + 1) + b(s^2 + 1) + (cs + d)s^2$$

$$= (a + c)s^3 + (b + d)s^2 + as + b$$

Equating like powers of s,

$$a + c = 1,\ b + d = 0,\ a = 0,\ b = 1$$

$$\therefore \quad a = 0,\ b = 1,\ c = 1,\ d = -1$$

$$\therefore \quad \bar{y} = \frac{1}{s^2} + \frac{s - 1}{s^2 + 1} = \frac{1}{s^2} + \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}$$

Taking inverse Laplace transform

$$y = L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{s}{s^2 + 1}\right) - L^{2}\left(\frac{1}{s^2 + 1}\right)$$

$$= t + \cos t - \sin t.$$

Ex. 14 : Solve by using Laplace transform

$(D^2 + 2D + 5)y = e^{-t}\sin t$, when $y(0) = 0$, $y'(0) = 1$. **(M.U. 1995, 2003)**

Sol. : Let $L(y) = \bar{y}$. Then taking Laplace transform of both sides,

$$L(y'') + 2L(y') + 5L(y) = L(e^{-t}\sin t)$$

But $\quad L(y') = s\bar{y} - y(0) = s\bar{y}$

And $\quad L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1$

And $\quad L\,e^{-t}\sin t = \dfrac{1}{(s+1)^2 + 1}$

$\therefore$ The equation becomes

$$(s^2\bar{y} - 1) + 2s\bar{y} + 5\bar{y} = \frac{1}{(s+1)^2 + 1}$$

$$\therefore \quad (s^2+2s+5)\,\bar{y} = 1 + \frac{1}{s^2+2s+2} = \frac{s^2+2s+3}{s^2+2s+2}$$

$$\therefore \quad \bar{y} = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

Let $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \dfrac{as+b}{(s^2+2s+5)} + \dfrac{cs+d}{(s^2+2s+2)}$

After simplification, we get

$$s^2+2s+s = (a+c)s^3 + (2a+b+2c+d)s^2 + (2a+2b+5c+2d)s + (2b+5d)$$

Equating the coefficients of like powers of s, we get,

$a+c=0,\ 2a+b+2c+d=1,$

$2a+2b+5c+2d=2,\ 2b+5d=3$

$$\therefore \quad a=0,\ b=\frac{2}{3},\ c=0,\ d=\frac{1}{3}$$

$$\therefore \quad \bar{y} = \frac{2}{3}\cdot\frac{1}{s^2+2s+5} + \frac{1}{3}\cdot\frac{1}{s^2+2s+2}$$

$$= \frac{2}{3}\cdot\frac{1}{(s+1)^2+2^2} + \frac{1}{3}\cdot\frac{1}{(s+1)^2+1^2}$$

Taking inverse Laplace transform

$$y = \frac{2}{3}\cdot e^{-t}\cdot L^{-1}\left[\frac{1}{s^2+2^2}\right] + \frac{1}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$= \frac{2}{3}e^{-t}\cdot\frac{1}{2}\sin 2t + \frac{1}{3}e^{-t}\sin t = \frac{e^{-t}}{3}(\sin 2t + \sin t).$$

Ex. 15 : Solve using Laplace transform $\dfrac{d^2y}{dt^2} + 9y = 18t$, given that $y(0)=0$ and $y(\pi/2)=0$. **(M.U. 1996, 98)**

Sol. : Let $L(y) = \bar{y}$. Then taking Laplace transforms of both sides

$$L(y'') + 9L(y) = 18L(t) \qquad \text{..........(1)}$$

But $L(y'') = s^2\bar{y} - sy(0) - y'(0)\ ;\ L(t) = \dfrac{1}{s^2}$

Since, $y'(0)$ is not given let us assume $y'(0) = A$. Hence, (1) becomes

$$s^2\bar{y} - A + 9\bar{y} = \frac{18}{s^2}$$

$$\therefore \quad (s^2+9)\,\bar{y} = \frac{18}{s^2} + A \qquad \therefore \quad \bar{y} = \frac{18}{s^2(s^2+9)} + \frac{A}{s^2+9}$$

$$\therefore \quad \bar{y} = \frac{18}{9}\left[\frac{1}{s^2} - \frac{1}{s^2+9}\right] + \frac{A}{s^2+9}$$

$$= \frac{2}{s^2} + \left(\frac{A-2}{s^2+9}\right)$$

$$\therefore \quad y = 2L^{-1}\left(\frac{1}{s^2}\right) + (A-2)\,L^{-1}\frac{1}{s^2+9}$$

$$\therefore \quad y = 2t + \frac{A-2}{3}\sin 3t$$

To find A we put $t = \dfrac{\pi}{2}$ and use that $y\left(\dfrac{\pi}{2}\right) = 0$.

$$\therefore \quad 0 = 2\cdot\frac{\pi}{2} + \left(\frac{A-2}{3}\right)\sin\left(\frac{3\pi}{2}\right) \qquad \therefore \quad 0 = \pi - \frac{(A-2)}{3}$$

$$\therefore \quad 0 = 3\pi - A + 2 \qquad \therefore \quad A = 3\pi + 2$$

$$\therefore \quad y = 2t + \pi\sin 3t.$$

Ex. 16 : Solve $(D^3 - 2D^2 + 5D)y = 0$, with $y(0)=0,\ y'(0)=0,\ y''(0)=1$.

Sol. : Let $\bar{y}$ be the Laplace transform of y *i.e.* let $L(y) = \bar{y}$.

Taking Laplace transform of both sides,

$$L(y''') - 2L(y'') + 5L(y') = 0$$

$$L(y') = s(\bar{y}) - y(0),\ L(y'') = s^2\bar{y} - s\,y(0) - y'(0)$$

$$L(y''') = s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)$$

From given conditions,

$$L(y') = s\bar{y},\ L(y'') = s^2\bar{y},\ L(y''') = s^3\bar{y} - 1.$$

$\therefore$ The equation becomes

$$s^3\bar{y} - 1 - 2s^2\bar{y} + 5s\,\bar{y} = 0 \qquad \therefore \quad \bar{y} = \frac{1}{s^3 - 2s^2 + 5s}$$

Taking inverse Laplace transform

$$y = L^{-1}\left[\frac{1}{s^3-2s^2+5s}\right] = L^{-1}\left[\frac{1}{s(s^2-2s+5)}\right]$$

$$= L^{-1}\left[\frac{1}{s[(s-1)^2+2^2]}\right]$$

We obtain the inverse by convolution theorem.

Let $\Phi_1(s) = \dfrac{1}{(s-1)^2+2^2}$ and $\Phi_2(s) = \dfrac{1}{s}$ $\quad\therefore\ \Phi(s) = \Phi_1(s)\cdot\Phi_2(s)$

$$\therefore \quad f_1(t) = L^{-1}\Phi_1(s) = L^{-1}\left[\frac{1}{(s-1)^2+2^2}\right]$$

$$= e^t \cdot L^{-1}\frac{1}{s^2+2^2} = \frac{1}{2}\cdot e^t \cdot \sin 2t$$

$$f_2(t) = L^{-1}\Phi_2(s) = L^{-1}\left(\frac{1}{s}\right) = 1$$

$$\therefore \quad f_1(u) = \frac{1}{2}e^u \sin 2u \text{ and } f_2(t-u) = 1$$

$$\therefore \quad L^{-1}\Phi(s) = \int_0^t \frac{1}{2}e^u \sin 2u\,du = \frac{1}{2}\cdot\frac{1}{5}\cdot\left[e^u(\sin 2u - 2\cos 2u)\right]_0^t$$

$$\therefore \quad y = \frac{1}{10}\left[e^t(\sin 2t - 2\cos 2t) + 2\right]$$

$\therefore$ The solution is $y = \frac{1}{5} - \frac{1}{5}e^t\cos 2t + \frac{1}{10}e^t \sin 2t.$

Ex. 17 : Solve the equation $y + \int_0^t y\,dt = 1 - e^{-t}$.

Sol. : Let $L(y) = \bar{y}$. Taking the Laplace transform of both sides, we get,

$$L(y) + L\left[\int_0^t y\,dt\right] = L(1) + L(e^{-t})$$

Since, $L\left[\int_0^t y\,dt\right] = \int_0^\infty e^{-st}\int_0^\infty y\,dt$

$$= \left[\int_0^t y\,dt \cdot \frac{e^{-st}}{s}\right]_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}\cdot y\,dt$$

$$= 0 + \frac{1}{s}\int_0^\infty e^{-st} y\,dt = \frac{1}{s}L(y) = \frac{1}{s}\bar{y}$$

and $L e^{-t} = \frac{1}{s+1}$, the equation becomes

$$\bar{y} + \frac{\bar{y}}{s} = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}$$

$$\therefore \quad \bar{y}\frac{(s+1)}{s} = \frac{1}{s(s+1)} \qquad \therefore \quad \bar{y} = \frac{1}{(s+1)^2}$$

$$\therefore \quad y = L^{-1}\frac{1}{(s+1)^2} = e^{-t}L^{-1}\frac{1}{s^2} = e^{-t}\cdot t$$

$$\therefore \quad y = t\,e^{-t}.$$

Ex. 18 : Solve the following equation by using Laplace transform

$\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t$, given that $y(0) = 1$. **(M.U. 1998, 99)**

Sol. : Let $L(y) = \bar{y}$. Taking Laplace transform of both sides, we get

$$L(y') + 2L(y) + L\left[\int_0^t y\,dt\right] = L(\sin t)$$

But $L(y') = sL(y) - y(0) = s\bar{y} - 1$

$$L\left[\int_0^t y\,dt\right] = \frac{1}{s}L(y) = \frac{1}{s}\bar{y}, \quad L(\sin t) = \frac{1}{s^2+1}$$

$\therefore$ The equation becomes

$$s\bar{y} - 1 + 2\bar{y} + \frac{1}{s}\bar{y} = \frac{1}{s^2+1}$$

$$\therefore \quad \left(s + 2 + \frac{1}{s}\right)\bar{y} = \frac{1}{s^2+1} + 1 = \frac{s^2+1+1}{s^2+1}$$

$$\therefore \quad \frac{(s^2+2s+1)}{s}\bar{y} = \frac{(s^2+2)}{s^2+1}$$

$$\therefore \quad \bar{y} = \frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

Let $\frac{s(s^2+2)}{(s+1)^2(s^2+1)} = \frac{a}{s+1} + \frac{b}{(s+1)^2} + \frac{cs+d}{s^2+1}$

$$\therefore \quad s(s^2+2) = a(s+1)(s^2+1) + b(s^2+1) + (cs+d)(s+1)^2$$

Putting $s = -1$, $-3 = 2b$ $\quad\therefore\ b = -3/2$

Putting $s = 0$, $0 = a + b + d$.

Equating the coefficients of s^2 and s^3.

$0 = a + b + 2c + d$ and $1 = a + c$

$\because\ b = -3/2, \quad a + d = 3/2$

and $a + 2c + d = 3/2$.

But $a + d = 3/2 \quad \therefore\ 2c = 0 \quad \therefore\ c = 0$

$\because\ 1 = a + c$ and $c = 0 \quad \therefore\ a = 1$

$\because\ a + d = 3/2$ and $a = 1 \quad \therefore\ d = 1/2$

$\therefore\ a = 1,\ b = -3/2,\ c = 0,\ d = 1/2$

$$\therefore \quad \bar{y} = \frac{1}{s+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$\therefore \quad y = L^{-1}\left(\frac{1}{s+1}\right) - \frac{3}{2}e^{-t}L^{-1}\frac{1}{s^2} + \frac{1}{2}L^{-1}\frac{1}{s^2+1}$$

$$\therefore \quad y = e^{-t} - \frac{3}{2} e^{-t} \cdot t + \frac{1}{2} \sin t.$$

Ex. 19 : Solve the equation $y(t) = kt + \int_0^t y(u) \cdot \sin(t-u)\, du$.

Sol. : We know that by convolution theorem, if $L f_1(t) = \Phi_1(s)$ and $L f_2(t) = \Phi_2(s)$, then

$$L^{-1}[\Phi_1(s) \cdot \Phi_2(s)] = \int_0^t f_1(u) \cdot f_2(t-u)\, du$$

i.e. $\Phi_1(s) \cdot \Phi_2(s) = L\left[\int_0^t f_1(u) \cdot f_2(t-u)\, du\right]$

i.e. $L\left[\int_0^t f_1(u) \cdot f_2(t-u)\, du\right] = L f_1(t) \cdot L f_2(t)$

Hence, taking Laplace transforms of both sides.

$$L\, y(t) = L(kt) + L\left[\int_0^t y(u) \sin(t-u)\, du\right]$$

$$\therefore \quad Ly(t) = L(kt) + L\, y(t) \cdot L \sin t$$

$$\therefore \quad \bar{y} = \frac{k}{s^2} + \bar{y} \cdot \frac{1}{s^2+1}$$

$$\therefore \quad \bar{y}\left(1 - \frac{1}{s^2+1}\right) = \frac{k}{s^2} \qquad \therefore \quad \bar{y} = k\frac{s^2+1}{s^4}$$

$$\therefore \quad \bar{y} = k\left[\frac{1}{s^2} + \frac{1}{s^4}\right]$$

$$\therefore \quad y = k\left[L^{-1}\frac{1}{s^2} + L^{-1}\frac{1}{s^4}\right] = k\left(\frac{1}{t} + \frac{t^3}{3!}\right)$$

Ex. 20 : Solve $\frac{d^2y}{dt^2} + 9y = \delta(t)$ given that $y = 0$, $\frac{dy}{dt} = 0$ at $t = 0$.

Sol. : By taking Laplace transforms of both sides.

$$s^2\bar{y} - sy(0) - y'(0) + 9\bar{y} = 1 \quad [\because \; L\,\delta(t) = 1]$$

$$\therefore \quad (s^2+9)\,\bar{y} = 1 \qquad \therefore \quad \bar{y} = \frac{1}{s^2+9}$$

$$\therefore \quad y = \frac{1}{3}\sin 3t.$$

Ex. 21 : Solve $\frac{d^2y}{dt^2} + \frac{3dy}{dt} + 2y = t\,\delta(t-1)$ with the conditions $y(0) = 0$, $y'(0) = 0$.

Sol. : Taking Laplace transforms of both sides

$$L(y'') + 3L(y') + 2L(y) = L[t\,\delta(t-1)]$$

But $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y}$

$L(y') = s\bar{y} - y(0) = s\bar{y}$

And $L[t\,\delta(t-1)] = e^{-s}(1)$ $\qquad [\because \; L f(t) \cdot \delta(t-a) = e^{-as} f(a)]$

$$\therefore \quad s^2\bar{y} + 3s\,\bar{y} + 2\bar{y} = e^{-s}$$

$$\therefore \quad (s^2+3s+2)\,\bar{y} = e^{-s}$$

$$\therefore \quad \bar{y} = \frac{e^{-s}}{s^2+3s+2} = \frac{e^{-s}}{(s+1)(s+2)} = e^{-s}\left[\frac{1}{s+1} - \frac{1}{s+2}\right]$$

Taking inverse Laplace transform

$$y = e^{-(t-1)} H(t-1) - e^{-2(t-1)} H(t-1).$$

Ex. 22 : Solve $\frac{d^2y}{dt^2} + 4y = f(t)$, with conditions $y(0) = 0$, $y'(0) = 1$

and $f(t) = \begin{cases} 1 & \text{when } 0 < t < 1 \\ 0 & \text{when } t > 1 \end{cases}$ **(M.U. 2005)**

Sol. : Taking Laplace transforms of both sides

$$L(y'') + 4L(y) = L\,f(t)$$

But $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1$

$\therefore \quad s^2\bar{y} - 1 + 4\bar{y} = \Phi(s)$ where, $L f(t) = \Phi(s)$

$$\therefore \quad (s^2+4)\,\bar{y} = 1 + \Phi(s)$$

$$\therefore \quad \bar{y} = \frac{1}{s^2+4} + \frac{\Phi(s)}{s^2+4}$$

We now express $f(t)$ as Heavisides unit step function

$$\therefore \quad f(t) = H(t) - H(t-1) \qquad \therefore \quad L f(t) = \frac{1}{s} - e^{-s} \cdot \frac{1}{s}.$$

$$\therefore \quad \bar{y} = \frac{1}{s^2+4} + \frac{1}{s^2+4}\left[\frac{1}{s} - e^{-s} \cdot \frac{1}{s}\right]$$

$$= \frac{1}{s^2+4} + \frac{1}{s(s^2+4)} - e^{-s}\frac{1}{s(s^2+4)}$$

Taking inverse Laplace transform

$$y = \frac{1}{2}\sin 2t + \frac{1}{4}(1 - \cos 2t) - \frac{1}{4}\{1 - \cos(t-1)\}\, H(t-1)$$

(For inverse see Ex. 1(i), page 4-45.)

Ex. 23 : Solve the above equation if $f(t) = H(t-2)$. **(M.U. 2002)**

Sol. : Since $f(t) = H(t-2)$

$$\Phi(s) = L\,f(t) = e^{-2s} \cdot \frac{1}{s}$$

$$\therefore \quad \bar{y} = \frac{1}{s^2+4} + e^{-2s} \cdot \frac{1}{s(s^2+4)}$$

$$= \frac{1}{s^2+4} + \frac{e^{-2s}}{4} \cdot \left[\frac{1}{s} - \frac{1}{s^2+4}\right]$$

Taking inverse of both sides

$$y = L^{-1}\left(\frac{1}{s^2+4}\right) + L^{-1}\left[\frac{e^{-2s}}{4} \cdot \frac{1}{s}\right] - L^{-1}\left[\frac{e^{-2s}}{4} \cdot \frac{1}{s^2+4}\right]$$

$$= \frac{1}{2}\sin 2t + \frac{1}{4} \cdot 1 \cdot H(t-2) - \frac{1}{4}\cos 2(t-2)\,H(t-2)$$

EXERCISE

Using Laplace transform solve the following differential equations with the given conditions.

1. $\frac{dy}{dt} + 2y = e^{-3t}$, $y(0) = 1$
2. $\frac{dy}{dt} + y = e^{-2t}$, $y(0) = 0$
3. $\frac{dy}{dt} + 2y = 5$, $y(0) = 1$
4. $\frac{dy}{dt} + 2y = \sin t$, $y(0) = 0$
5. $\frac{dy}{dt} + y = \cos 2t$, $y(0) = 1$
6. $(D-2)\,x + 3y = 0$
   $2x + (D-1)\,y = 0$, $x(0) = 8$, $y(0) = 3$.
7. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$; at $x = 0$, $y = 0$, $\frac{dy}{dx} = 4$.
8. $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} = -8t$; $x(0) = 0 = x'(0)$.
9. $(D^2 + 9)\,y = 18t$; $y(0) = 0$, $y'(0) = 0$.
10. $f''(t) + f'(t) = t$; $f(0) = 1$, $f'(0) = -1$.
11. $(D^2 - 3D + 2)\,y = 2e^{3t}$; $y = 2$, $y' = 3$ at $t = 0$.
12. $(D^2 + 1)\,y = \sin t$; $y(0) = 1$, $y'(0) = -\frac{1}{2}$.
13. $(D^2 - 2D + 2)\,x = 0$; $x(0) = x'(0) = 1$.
14. $y'' - 2y' + y = e^t$; $y(0) = 2$, $y'(0) = -1$.

15. $(D^2 + D - 2)\,x = 2(1 + t - t^2)$; $x = 0$, $Dx = 3$ for $t = 0$.
16. $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 2$; $y(0) = 1$, $y'(0) = 0$.
17. $(D^2 - 3D + 2)\,y = 4t + e^{3t}$ if $y = 1$, $Dy = -1$ at $t = 0$.
18. $(D^3 + 2D^2 - D - 2)\,y = 0$ if $y = 1$, $Dy = 2$, $D^2y = 0$ at $y = 0$.
19. $(D^2 + 4D + 3)\,y = e^{-t}$; $y(0) = y'(0) = 1$.
20. $(D^2 + D)\,y = t^2 + 2t$ at $t = 0$, $y = 4$ and $Dy = 2$. **(M.U. 1997)**
21. $(D^2 - 2D + 1)\,x = e^t$ with the conditions $x = 2$, $Dx = -1$ at $t = 0$. **(M.U. 1997, 2000)**
22. $(D+1)^2\,y = 6t\,e^{-t}$ with $y(0) = 2$, $y'(0) = 5$. **(M.U. 1994)**
23. $y(t) = a\sin t - 2\int_0^t y(u)\cos(t-u)\,du$.
24. $y(t) = 4t - 3\int_0^t y(u)\sin(t-u)\,du$.
25. $\frac{d^2y}{dx^2} + 16y = \delta(t)$ given that $y = 0$, $\frac{dy}{dt} = 0$ at $t = 0$.
26. $(D^2 - 3D + 2)\,y = 4e^{2t}$ at $t = 0$, $y = -3$ and $Dy = 5$. **(M.U. 2004)**
27. $(D^2 - 2D - 8)\,y = 4$, $y(0) = 0$ and $y'(0) = 1$. **(M.U. 2003, 04)**
28. $(3D + 2)\,y = e^{3t}$, $y(0) = 1$. **(M.U. 2002)**
29. $(D^2 + 2D + 1)\,y = 3\,t\,e^{-t}$, $y(0) = 4$, $y'(0) = 2$. **(M.U. 2002)**
30. $(D^2 + D)\,y = t^2 + 2t$, $y(0) = 4$, $y'(0) = -2$. **(M.U. 2004)**
31. $(D^2 - 4)\,y = 3e^t$, $y(0) = 0$, $y'(0) = 3$. **(M.U. 2004)**

[**Ans. : (1)** $y = 2e^{-2t} - e^{-3t}$, **(2)** $y = e^{-t} - e^{-2t}$, **(3)** $y = \frac{5}{2} - \frac{3}{2}e^{-2t}$,

(4) $y = -\frac{1}{5}\cos t + \frac{2}{5}\sin t + e^{-2t}$, **(5)** $y = \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t + \frac{4}{5}e^{-t}$,

(6) $x = 3e^{4t} + 5e^{-t}$, $y = -2e^{4t} + 5e^{-t}$, **(7)** $y = e^x - e^{-3x}$,

(8) $y = -\frac{1}{8} + \frac{1}{2}t - t^2 + \frac{e^{-4t}}{8}$, **(9)** $y = 2\left[t - \frac{1}{3}\sin 3t\right]$,

(10) $x = 1 - t + \frac{t^2}{2}$, **(11)** $y = 2e^t - e^{2t} + e^{3t}$,

(12) $y = \left(1 - \frac{t}{2}\right)\cos t$, **(13)** $x = e^t\cos t$,

(14) $y = e^t\left(2 - 3t + \dfrac{t^2}{2}\right)$, (15) $x = t^2 + e^t - e^{-2t}$,

(16) $y = -\dfrac{1}{3} + \dfrac{8}{15}e^{3t} + \dfrac{4}{5}e^{-2t}$, (17) $y = -\dfrac{1}{2}e^t - 2e^{2t} + \dfrac{1}{2}e^{3t} + 2t + 3$,

(18) $y = \dfrac{1}{3}(5e^t + e^{-2t}) - e^{-t}$, (19) $y = \dfrac{7}{4}e^{-t} + \dfrac{3}{4}e^{-3t} + \dfrac{1}{2}t\,e^{-t}$,

(20) $y = \dfrac{t^3}{3} + 2e^{-t} + 2$, (21) $x = 2e^t - 3e^t \cdot t + \dfrac{t^2}{2}e^t$,

(22) $y = e^{-t}(t^3 + 7t + 2)$, (23) $y = a\,t\,e^{-t}$,

(24) $y = t + \dfrac{3}{2}\sin 2t$, (25) $y = \dfrac{1}{4}\sin 4t$,

(26) $y = -7e^t + 4e^{2t} + 4t\,e^{2t}$,

(27) $y = \dfrac{1}{6}[e^{4t} + 2e^{-2t} - 3 + 2e^t \sin h\, 3t]$,

(28) $y = \dfrac{1}{11}(e^{3t} + 10\,e^{-2t/3})$, (29) $y = e^{-t}\left(4 + 6t + \dfrac{t^3}{2}\right)$,

(30) $y = 2 + 2e^{-t} + \dfrac{t^3}{3}$, (31) $y = -e^t + \dfrac{3}{2}e^{2t} - \dfrac{1}{2}e^{-2t}$.]

Theory

1. State and prove convolution theorem. **(M.U. 2003)**
2. State convolution theorem and deduce that

$$L^{-1}\left[\frac{1}{s}\Phi(s)\right] = \int_0^t f(u)\,du \quad \text{where, } L[f(u)] = \Phi(s).$$

3. Find $L[\delta(t-a)]$ where,

$$\delta(t-a)\begin{cases} = 0 & \text{for } t < a \\ = 1/\varepsilon & \text{for } a < t < a + \varepsilon. \\ = 0 & \text{for } t > a + \varepsilon \end{cases}$$

(M.U. 1998)

4. If $f(t-a) \cdot H(t-a) = 0 \quad t < a$

$= f(t-a) \quad t > a$, then prove that

$$L[f(t-a) \cdot H(t-a)] = e^{-as}L[f(t)] = e^{-as}\Phi(s)$$

where, $\Phi(s) = L[f(t)]$. **(M.U. 1996, 2004)**

5. If $f(t)$ is a periodic function of period a, prove that

$$L[f(t)] = \frac{1}{1 - e^{-as}}\int_0^a e^{-st} f(t)\,dt.$$

(M.U. 1995, 96, 97, 2003)

6. Define Heaviside's unit step function $H(t-a)$ and obtain its Laplace transform.
7. Define Heaviside unit step function and obtain Laplace transform of $f(t-a) \cdot H(t-a)$.
8. Define Dirac-delta function and obtain its Laplace transform. **(M.U. 2004)**

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